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AN EXPLANATION  
OF THE  
OBSERVED IRREGULARITIES  
IN THE  
MOTION OF URANUS,

ON THE HYPOTHESIS OF DISTURBANCES  
CAUSED BY A MORE DISTANT PLANET;  
WITH A DETERMINATION OF THE MASS, ORBIT, AND POSITION  
OF THE DISTURBING BODY.

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*(From the Appendix to the Nautical Almanac for the Year 1851.)*

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1846.

## ADVERTISEMENT.

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THIS Paper was communicated by the Author to the Royal Astronomical Society, and was read to that body, at their ordinary meeting, on November 13, 1846. The Press of the Society being engaged on an extensive paper, on the longitude of Valentia, by the Astronomer Royal, and it being deemed of national importance that Mr. ADAMS's Paper should be submitted to the world without loss of time, application was made to Capt. W. H. SMYTH, R.N., President, and to the Rev. R. SHEEPSHANKS, Secretary, of the Society, who, with their usual promptitude and zeal, granted permission for the immediate printing and publishing of the Paper by the NAUTICAL ALMANAC OFFICE; and it is under these circumstances that the investigations of Mr. ADAMS first appear as an extract from the Appendix to the NAUTICAL ALMANAC for 1851.

W. S. STRATFORD,  
Superintendent of the Nautical Almanac.

*Nautical Almanac Office,  
3, Verulam Buildings, Gray's Inn, London.  
December 31, 1846.*

## ON THE PERTURBATIONS OF URANUS.

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1. THE irregularities in the motions of Uranus have for a long time engaged the attention of Astronomers. When the path of the planet became approximately known, it was found that, previously to its discovery by Sir W. Herschel in 1781, it had several times been observed as a fixed star by Flamsteed, Bradley, Mayer, and Lemonnier. Although these observations are doubtless very far inferior in accuracy to the modern ones, they must be considered valuable, in consequence of the great extension which they give to the observed arc of the planet's orbit. Bouvard, however, to whom we owe the Tables of Uranus at present in use, found that it was impossible to satisfy these observations, without attributing much larger errors to the modern observations than they admit of, and consequently founded his Tables exclusively on the latter. But in a very few years sensible errors began again to show themselves, and though the Tables were formed so recently as 1821, their error at the present time exceeds two minutes of space, and is still rapidly increasing. There appeared, therefore, no longer any sufficient reason for rejecting the ancient observations, especially since, with the exception of Flamsteed's first observation, which is more than twenty years anterior to any of the others, they are mutually confirmatory of each other.

2. Now that the discovery of another planet has confirmed in the most brilliant manner the conclusions of analysis, and enabled us with certainty to refer these irregularities to their true cause, it is unnecessary for me to enter at length upon the reasons which led me to reject the various other hypotheses which had been formed to account for them. It is sufficient to say, that they all appeared to be very improbable in themselves, and incapable of being tested by any exact calculation. Some had even supposed that at the great distance of Uranus from the Sun, the law of attraction becomes different from that of the inverse square of the distance. But the law of gravitation was too firmly established for this to be admitted, till every other hypothesis had failed, and I felt convinced that in this, as in every previous instance of the kind, the discrepancies which had for a time thrown doubts on the truth of the law, would eventually afford the most striking confirmation of it.

3. My attention was first directed to this subject several years since, by reading Mr. Airy's valuable Report on the recent progress of Astronomy. I find among my



papers the following memorandum, dated July 3, 1841:—"Formed a design, in the beginning of this week, of investigating, as soon as possible after taking my degree, the irregularities in the motion of Uranus which are yet unaccounted for; in order to find whether they may be attributed to the action of an undiscovered planet beyond it, and if possible, thence to determine approximately the elements of its orbit, &c., which would probably lead to its discovery." Accordingly, in 1843, I attempted a first solution of the problem, assuming the orbit to be a circle, with a radius equal to twice the mean distance of Uranus from the Sun. Some assumption as to the mean distance was clearly necessary in the first instance, and Bode's law appeared to render it probable that the above would not be far from the truth. This investigation was founded exclusively on the modern observations, and the errors of the Tables were taken from those given in the Equations of Condition of Bouvard's Tables as far as the year 1821, and subsequently from the observations given in the *Astronomische Nachrichten*, and from the Cambridge and Greenwich Observations. The result showed that a good general agreement between theory and observation might be obtained; but the larger differences occurring in years where the observations used were deficient in number, and the Greenwich Planetary Observations being then in process of reduction, I applied to Mr. Airy, through the kind intervention of Professor Challis, for the observations of some years in which the agreement appeared least satisfactory. The Astronomer Royal, in the kindest possible manner, sent me, in February 1844, the results of all the Greenwich Observations of Uranus.

4. Meanwhile the Royal Academy of Sciences of Göttingen had proposed the Theory of Uranus as the subject of their mathematical prize, and although the little time which I could spare from important duties in my college prevented me from attempting the complete examination of the theory, which a competition for the prize would have required, yet this fact, together with the possession of such a valuable series of observations, induced me to undertake a new solution of the problem. I now took into account the most important terms depending on the first power of the eccentricity of the disturbing planet, retaining the same assumption as before with respect to the mean distance. For the modern observations, the errors of the Tables were taken exclusively from the Greenwich Observations as far as the year 1830, with the exception of an observation by Bessel, in 1823; and subsequently from the Cambridge and Greenwich Observations, and those given in various numbers of the *Astronomische Nachrichten*. The errors of the Tables for the ancient Observations were taken from those given in the Equations of Condition of Bouvard's Tables. After obtaining several solutions differing little from each other, by gradually taking into account more and more terms of the series expressing the Perturbations, I communicated to Professor Challis, in September 1845, the final values which I had obtained for the mass, heliocentric longitude, and elements of the orbit of the assumed planet. The same results, slightly corrected, I communicated in the following month to the Astronomer Royal. The eccentricity coming out much larger than was probable, and later observations showing that the theory founded on the first hypothesis as to the mean distance, was still sensibly in error, I afterwards repeated my investigation, supposing the mean distance to be about  $\frac{1}{80}$ th part less than before. The result,

which I communicated to Mr. Airy, in the beginning of September of the present year, appeared more satisfactory than my former one, the eccentricity being smaller, and the errors of theory, compared with late observations, being less, and led me to infer that the distance should be still further diminished.

5. In November, 1845, M. Le Verrier presented to the Royal Academy of Sciences at Paris, a very complete and elaborate investigation of the Theory of Uranus, as disturbed by the action of Jupiter and Saturn, in which he pointed out several small inequalities which had previously been neglected; and in June, of the present year, he followed up this investigation by a memoir, in which he attributed the residual disturbances to the action of another planet at a distance from the Sun equal to twice that of Uranus, and found a longitude for the new planet agreeing very nearly with the result which I had obtained on the same hypothesis. On the 31st of August he presented to the Academy a more complete investigation, in which he determined the mass and the elements of the orbit of the new planet, and also obtained limiting values of the mean distance and heliocentric longitude. I mention these dates merely to show that my results were arrived at independently, and previously to the publication of those of M. Le Verrier, and not with the intention of interfering with his just claims to the honours of the discovery; for there is no doubt that his researches were first published to the world, and led to the actual discovery of the planet by Dr. Galle, so that the facts stated above cannot detract, in the slightest degree, from the credit due to M. Le Verrier.

6. In order not to have an inconvenient number of equations of condition, I divided the modern observations into groups, each including a period of three years, and as Mr. Airy had shown that the error of the Tabular Radius Vector was sometimes considerable, I either selected those observations which were made near opposition, or combined the others in such a manner that the result should be nearly free from the effects of this error. From the observations of each group, the error of the Tables in heliocentric longitude was found, corresponding to the time of mean opposition in the middle year of the group. Thus were formed 21 normal errors of the Tables, corresponding to as many equidistant periods between 1780 and 1840. The error for 1780 was found by interpolating between the errors of 1781, 1782, and 1783, and those given by the Ancient Observations of 1769 and 1771, and though not entitled to the same weight as the others, cannot, I think, be liable to much uncertainty. In my last calculations, I might have used more recent observations, but in order to obtain the effect due to the change of mean distance, it was necessary that the investigation should be founded on the same elements as before, and the later observations might be used as a test of the theory.

7. In order to satisfy myself that there was no important error in Bouvard's Tables, I recomputed all the principal inequalities produced by the action of Jupiter and Saturn, and found no difference of any consequence except in the equation depending on the mean longitude of Saturn minus twice that of Uranus, the error of which had been already pointed out by Bessel. The principal equation depending on the action of Jupiter, also required correction in consequence of the increased value which has been lately obtained for the mass of that planet. The corrections to be applied to Bouvard's Tables on these accounts, are the following :

$$+ 1^{\circ} 9' 18'' \sin \{ \phi_1 - 2\phi_2 - 13^{\circ} 1' 5'' \} \\ + 1^{\circ} 0' 85'' \sin \{ \phi - \phi_2 \}$$

$\phi$ ,  $\phi_1$ ,  $\phi_2$  being the mean longitudes of Jupiter, Saturn, and Uranus, respectively. In the Reduction of the Greenwich Observations, the latter correction was already taken into account. M. Hansen having also found some new inequalities in the motion of Uranus, depending on the square of the disturbing force, I re-computed the values of these, following the same method as that given by M. Delaunay in the *Conn. des Temps* for 1845, and my results agreed very closely with his, the terms to be added to the longitude being

$$+ 32^{\circ} 0' 0'' \sin \{ 3\phi_2 - 6\phi_1 + 2\phi + 22^{\circ} 18' 8'' \} \\ - 8^{\circ} 35' \sin \{ 2\phi_2 - 6\phi_1 + 2\phi + 39^{\circ} 10' 5'' \} \\ - 1^{\circ} 49' \sin \{ 4\phi_2 - 6\phi_1 + 2\phi + 34^{\circ} 48' 4'' \}$$

With respect to the inequalities of higher orders neglected by Bouvard, I considered that the most important of them would be, either those of long period, or those whose period was nearly equal to that of Uranus. During three-fourths of a revolution of the planet, the effects of the former class would be nearly confounded with those arising from a change in the epoch and mean motion, and those of the latter class with the effects produced by a constant change in the eccentricity and longitude of the Perihelion. The position of the planet to be determined would therefore be little affected by these terms, and the others would probably be much smaller than those which would necessarily be neglected in a first approximation to the perturbations produced by the new planet.

8. Taking into account the several corrections above-mentioned, the residual differences between the theoretical and observed heliocentric longitudes were the following:

<i>Ancient Observations.</i>		<i>Modern Observations.</i>	
Year.	Observation—Theory.	Year.	Observation—Theory.
1690	+ 61 .2	1780	+ 3 .46
1712	+ 92 .7	1783	+ 8 .45
1715	+ 73 .8	1786	+ 12 .36
1750	— 47 .6	1789	+ 19 .02
1753	— 39 .5	1792	+ 18 .70
1756	— 45 .7	1795	+ 21 .38
1764	— 34 .9	1798	+ 20 .95
1769	— 19 .3	1801	+ 22 .21
1771	— 2 .3	1804	+ 24 .16
		1807	+ 22 .07
		1810	+ 23 .16
		1813	+ 22 .00
		1816	+ 22 .88
		1819	+ 20 .69
		1822	+ 20 .97
		1825	+ 18 .16
		1828	+ 10 .82
		1831	— 3 .98
		1834	— 20 .80
		1837	— 42 .66
		1840	— 66 .64

9. It is easily seen that the series expressing the correction of the *Mean* longitude in terms of the corrections applied to the elements of the orbit, is more convergent than that which gives the correction of the *true* longitude, and the same thing is true for the perturbations of the mean longitude, as compared with those of the true. The corrections found above were accordingly converted into corrections of mean longitude by multiplying each of them by the factor  $\frac{r^2}{ab}$ ,  $r$  being the Rad. Vector, and  $a$  and  $b$  the semi-axes of the orbit. Hence these latter corrections were found to be the following :

*Ancient Observations.*

Year.	Observation—Theory. "
1690	+62.6
1712	+84.5
1715	+67.2
1750	—51.8
1753	—43.2
1756	—50.1
1764	—37.8
1769	—20.5
1771	— 2.4

*Modern Observations.*

Year.	Observation—Theory. "
1780	+ 3.42
1783	+ 8.19
1786	+11.74
1789	+17.75
1792	+17.22
1795	+19.52
1798	+19.06
1801	+20.24
1804	+22.19
1807	+20.52
1810	+21.89
1813	+21.19
1816	+22.50
1819	+20.78
1822	+21.50
1825	+18.97
1828	+11.50
1831	— 4.29
1834	—22.63
1837	—46.70
1840	—73.09

These numbers form the basis of the subsequent investigations.

10. Let  $\delta\epsilon$ ,  $\delta a$ ,  $\delta e$ , and  $\delta\omega$  denote the corrections to be applied to the Tabular Elements of Uranus, then the correction of the mean longitude at any time  $t$  is

$$= \delta\epsilon + 2e^2\delta\omega + t\delta n - \left\{ 2\cos(n t + \epsilon - \omega) + \frac{e}{2}\cos 2(n t + \epsilon - \omega) \right\} e\delta\omega \\ + \left\{ 2\sin(n t + \epsilon - \omega) + \frac{e}{2}\sin 2(n t + \epsilon - \omega) \right\} \delta e$$

If we include the small term  $2e^2\delta\omega$  in the quantity  $\delta\epsilon$ , this correction may be put under the following form :

$\delta\epsilon + t\delta n + \cos n t \delta x_1 + \sin n t \delta y_1 + \cos 2 n t \delta x_2 + \sin 2 n t \delta y_2$   
in which expression

$$\delta x_2 = \frac{1}{4}e \{ \cos(\epsilon - \omega) \delta x_1 + \sin(\epsilon - \omega) \delta y_1 \}$$

$$\delta y_2 = -\frac{1}{4}e \{ \sin(\epsilon - \omega) \delta x_1 - \cos(\epsilon - \omega) \delta y_1 \}$$

11. Also, adopting the notation of Pontécoulant's "Théorie Analytique," the perturbations of mean longitude

$$\begin{aligned}
 &= \frac{m'}{2} \sum F_i \sin i (n t - n' t + \varepsilon - \varepsilon') \\
 &\quad + m' e \sum G_i \sin \{ i (n t - n' t + \varepsilon - \varepsilon') - (n t + \varepsilon - \omega) \} \\
 &\quad + m' e' \sum H_i \sin \{ i (n t - n' t + \varepsilon - \varepsilon') - (n t + \varepsilon - \omega') \}
 \end{aligned}$$

Where the accented letters belong to the disturbing planet,  $i$  takes all integral values, positive and negative, except zero, and if we put  $i (n - n') = z$ , the values of  $F_i$ ,  $G_i$ , and  $H_i$  are the following :

$$\begin{aligned}
 F_i &= \left\{ \frac{3 i n^4}{z^3 (z^2 - n^2)} + \frac{i n^2}{z^2 - n^2} \right\} a A_i + \frac{2 n^2}{z (z^2 - n^2)} a^2 \frac{d A_i}{d a} \\
 G_i &= \left\{ -\frac{3 i (i-1) n^4}{(z-n)^2 z (z-2n)} - \frac{i (i+1) n^2}{z (z-2n)} + \frac{i n^2}{z^2 - n^2} + \frac{3 i n^2}{z (z-n) (z-2n)} \right\} a A_i \\
 &\quad + \left\{ -\frac{3}{2} \frac{(i-1) n^4}{(z-n)^2 z (z-2n)} - \frac{1}{2} \frac{(i-1) n^2}{z (z-2n)} - \frac{1}{2} \frac{n^2}{z^2 - n^2} - \frac{2 i n^2}{z (z-n) (z-2n)} \right\} a^2 \frac{d A_i}{d a} \\
 &\quad - \frac{n^2}{z (z-n) (z-2n)} a^2 \frac{d^2 A_i}{d a^2} \\
 H_i &= \left\{ \frac{3}{2} \frac{(i-1) (2i-1) n^4}{(z-n)^2 z (z-2n)} + \frac{1}{2} \frac{(i-1) (2i-1) n^2}{z (z-2n)} \right\} a A_{i-1} \\
 &\quad + \left\{ \frac{3}{2} \frac{(i-1) n^4}{(z-n)^2 z (z-2n)} + \frac{1}{2} \frac{(i-1) n^2}{z (z-2n)} + \frac{2 i n^2}{z (z-n) (z-2n)} \right\} a^2 \frac{d A_{i-1}}{d a} \\
 &\quad + \frac{n^2}{z (z-n) (z-2n)} a^2 \frac{d^2 A_{i-1}}{d a^2}
 \end{aligned}$$

12. Now, if we assume  $\frac{a}{\alpha}$  or  $\alpha = \sin 30^\circ = 0.5$ , the values of the fundamental

quantities  $b$ ,  $\alpha \frac{db}{d\alpha}$ ,  $\alpha^2 \frac{d^2 b}{d\alpha^2}$ , will be

$$\log. b_0 = 0.33170; \quad \log. \alpha \frac{db_0}{d\alpha} = 9.53765 \quad \log. \alpha^2 \frac{d^2 b_0}{d\alpha^2} = 9.77848$$

$$\log. b_1 = 9.74497; \quad \log. \alpha \frac{db_1}{d\alpha} = 9.83868; \quad \log. \alpha^2 \frac{d^2 b_1}{d\alpha^2} = 9.70857$$

$$\log. b_2 = 9.32425; \quad \log. \alpha \frac{db_2}{d\alpha} = 9.68012 \quad \log. \alpha^2 \frac{d^2 b_2}{d\alpha^2} = 9.87776$$

$$\log. b_3 = 8.94670; \quad \log. \alpha \frac{db_3}{d\alpha} = 9.46315 \quad \log. \alpha^2 \frac{d^2 b_3}{d\alpha^2} = 9.86253$$

Hence the principal inequalities of mean longitude, produced by the action of a planet whose mass is  $\frac{m'}{5000}$ , that of the Sun being unity, and the eccentricity of whose orbit is  $\frac{e'}{20}$  will be the following :

$$\begin{aligned} & -36^{\circ} 99' m' \sin \{n t - n' t + \varepsilon - \varepsilon'\} \\ & + 58^{\circ} 97' m' \sin 2 \{n t - n' t + \varepsilon - \varepsilon'\} \\ & + 5^{\circ} 80' m' \sin 3 \{n t - n' t + \varepsilon - \varepsilon'\} \\ & + 2^{\circ} 06' m' \sin \{n' t + \varepsilon' - \omega\} \\ & - 4^{\circ} 30' m' e' \sin \{n' t + \varepsilon' - \omega'\} \\ & + 31^{\circ} 25' m' \sin \{n t - 2n' t + \varepsilon - 2\varepsilon' + \omega\} \\ & - 12^{\circ} 14' m' e' \sin \{n t - 2n' t + \varepsilon - 2\varepsilon' + \omega'\} \\ & + 48^{\circ} 55' m' \sin \{2n t - 3n' t + 2\varepsilon - 3\varepsilon' + \omega\} \\ & - 93^{\circ} 01' m' e' \sin \{2n t - 3n' t + 2\varepsilon - 3\varepsilon' + \omega'\} \end{aligned}$$

To these may be added the following, which are of two dimensions in terms of the eccentricities :

$$\begin{aligned} & + 0^{\circ} 57' m' \sin 3 \{n t - n' t + \varepsilon - \varepsilon'\} \\ & - 1^{\circ} 08' m' e' \sin \{3(n t - n' t + \varepsilon - \varepsilon') - \omega + \omega'\} \end{aligned}$$

These expressions may be put under the following form :

$$\begin{aligned} & h_1 \cos (n - n') t + h_2 \cos 2 (n - n') t + h_3 \cos 3 (n - n') t \\ & + k_1 \sin (n - n') t + k_2 \sin 2 (n - n') t + k_3 \sin 3 (n - n') t \\ & + p_1 \cos n' t + p_2 \cos (n - 2n') t + p_3 \cos (2n - 3n') t \\ & + q_1 \sin n' t + q_2 \sin (n - 2n') t + q_3 \sin (2n - 3n') t \end{aligned}$$

13. Let the time of the mean opposition in 1810 be taken as the epoch from which  $t$  is reckoned ; this date, expressed in decimal parts of a year, will be 1810.328. Also, let 3 synodic periods of Uranus, = 3.0362 years, be taken for the unit of time ; then the change of the mean anomaly in an unit of time will be  $13^{\circ} 0' 5''$  ; also  $n = 13^{\circ} 0' 6''$ ,  $n' = 4^{\circ} 36' 0'' \therefore n - n' = 8^{\circ} 24' 6''$ ,  $n - 2n' = 3^{\circ} 48' 6''$ ,  $2n - 3n' = 12^{\circ} 13' 2''$ . Hence the equations of condition given by the modern observations will be of the form

$$\begin{aligned} c'' = & \delta \varepsilon + \delta x_1 \cos \{13^{\circ} 0' 5''\} t + \delta x_2 \cos \{26^{\circ} 1' 0''\} t \\ & + t \delta n + \delta y_1 \sin \{13^{\circ} 0' 5''\} t + \delta y_2 \sin \{26^{\circ} 1' 0''\} t \\ & + h_1 \cos \{8^{\circ} 24' 6''\} t + h_2 \cos \{16^{\circ} 49' 2''\} t + h_3 \cos \{25^{\circ} 13' 8''\} t \\ & + k_1 \sin \{8^{\circ} 24' 6''\} t + k_2 \sin \{16^{\circ} 49' 2''\} t + k_3 \sin \{25^{\circ} 13' 8''\} t \\ & + p_1 \cos \{4^{\circ} 36' 0''\} t + p_2 \cos \{3^{\circ} 48' 6''\} t + p_3 \cos \{12^{\circ} 13' 2''\} t \\ & + q_1 \sin \{4^{\circ} 36' 0''\} t + q_2 \sin \{3^{\circ} 48' 6''\} t + q_3 \sin \{12^{\circ} 13' 2''\} t \end{aligned}$$

in which  $t$  assumes all integral values from  $-10$  to  $+10$  in succession, and the several values of  $c''$  are contained in the table given in Article 9.

14. The final equations for the corrections of the elliptic elements will be found by multiplying each equation successively by the co-efficients of  $\delta\epsilon$ ,  $\delta n$ ,  $\delta x_1$  and  $\delta y_1$ , which occur in it, and adding the several results.

Let the equations be treated in a similar manner with reference to the quantities  $h_1, k_1, h_2, k_2, h_3, k_3, p_1, q_1, p_2, q_2$ .

It will be seen that, in consequence of the arrangement which has been given to the equations of condition, the equations thus formed naturally separate themselves into two groups, one of which involves only  $\delta\epsilon$ ,  $\delta x_1$ ,  $\delta x_2$ , with the quantities  $h$  and  $p$ , while the other involves  $\delta n$ ,  $\delta y_1$ ,  $\delta y_2$ , with the quantities  $k$  and  $q$ .

Also, the co-efficients in these equations are easily calculated by the following formulæ, putting  $t=10$  in their right hand members :

$$\begin{aligned}\Sigma 2 \cos m t &= \frac{\sin m (t + \frac{1}{2})}{\sin \frac{1}{2} m} \\ \Sigma 2 t \sin m t &= \frac{(t+1) \sin m t - t \sin m (t+1)}{2 \sin^2 \frac{1}{2} m} \\ \Sigma 2 \cos m t \cos n t &= \frac{1}{2} \left\{ \frac{\sin (m-n) (t + \frac{1}{2})}{\sin \frac{1}{2} (m-n)} + \frac{\sin (m+n) (t + \frac{1}{2})}{\sin \frac{1}{2} (m+n)} \right\} \\ \Sigma 2 \sin m t \sin n t &= \frac{1}{2} \left\{ \frac{\sin (m-n) (t + \frac{1}{2})}{\sin \frac{1}{2} (m-n)} - \frac{\sin (m+n) (t + \frac{1}{2})}{\sin \frac{1}{2} (m+n)} \right\} \\ \Sigma 2 \cos^2 m t &= t + \frac{1}{2} + \frac{1 \sin m (2t+1)}{2 \sin m} \\ \Sigma 2 \sin^2 m t &= t + \frac{1}{2} - \frac{1 \sin m (2t+1)}{2 \sin m}\end{aligned}$$

15. By performing the calculations, the equations of the first group are found to be the following :

$$\begin{aligned}(\epsilon) \quad 151 \cdot 48 &= 21 \cdot 0000 \delta\epsilon + 6 \cdot 0670 \delta x_1 - 4 \cdot 4358 \delta x_2 \\ &\quad + 13 \cdot 6320 h_1 + 0 \cdot 4043 h_2 - 4 \cdot 5608 h_3 \\ &\quad + 18 \cdot 6046 p_1 + 19 \cdot 3384 p_2 + 7 \cdot 3721 p_3 \\ (x) \quad 246 \cdot 48 &= 6 \cdot 0670 \delta\epsilon + 8 \cdot 2821 \delta x_1 + 4 \cdot 1762 \delta x_2 \\ &\quad + 7 \cdot 4041 h_1 + 8 \cdot 2523 h_2 + 4 \cdot 6963 h_3 \\ &\quad + 6 \cdot 5389 p_1 + 6 \cdot 3978 p_2 + 8 \cdot 1831 p_3 \\ (h_1) \quad 209 \cdot 74 &= 13 \cdot 6320 \delta\epsilon + 7 \cdot 4041 \delta x_1 - 0 \cdot 2337 \delta x_2 \\ &\quad + 10 \cdot 7022 h_1 + 4 \cdot 5356 h_2 - 0 \cdot 0018 h_3 \\ &\quad + 12 \cdot 7013 p_1 + 12 \cdot 9883 p_2 + 8 \cdot 0038 p_3 \\ (h_2) \quad 242 \cdot 68 &= 0 \cdot 4043 \delta\epsilon + 8 \cdot 2523 \delta x_1 + 7 \cdot 5650 \delta x_2 \\ &\quad + 4 \cdot 5356 h_1 + 10 \cdot 2960 h_2 + 8 \cdot 1944 h_3 \\ &\quad + 1 \cdot 7866 p_1 + 1 \cdot 3667 p_2 + 7 \cdot 6671 p_3 \\ (h_3) \quad 86 \cdot 67 &= - 4 \cdot 5608 \delta\epsilon + 4 \cdot 6963 \delta x_1 + 10 \cdot 5023 \delta x_2 \\ &\quad - 0 \cdot 0018 h_1 + 8 \cdot 1944 h_2 + 10 \cdot 7071 h_3 \\ &\quad - 3 \cdot 0812 p_1 - 3 \cdot 5347 p_2 + 3 \cdot 8855 p_3\end{aligned}$$

$$\begin{aligned}
 (p_2) \quad 165 \cdot 99 &= 19 \cdot 3384 \delta \epsilon + 6 \cdot 3978 \delta x_1 - 3 \cdot 4948 \delta x_2 \\
 &\quad + 12 \cdot 9883 h_1 + 1 \cdot 3667 h_2 - 3 \cdot 5347 h_3 \\
 &\quad + 17 \cdot 2795 p_1 + 17 \cdot 9106 p_2 + 7 \cdot 5423 p_3 \\
 (p_3) \quad 242 \cdot 56 &= 7 \cdot 3721 \delta \epsilon + 8 \cdot 1831 \delta x_1 + 3 \cdot 4071 \delta x_2 \\
 &\quad + 8 \cdot 0038 h_1 + 7 \cdot 6671 h_2 + 3 \cdot 8855 h_3 \\
 &\quad + 7 \cdot 6127 p_1 + 7 \cdot 5423 p_2 + 8 \cdot 2019 p_3
 \end{aligned}$$

16. By means of ( $\epsilon$ ) eliminate  $\delta \epsilon$  from each of the other equations, and these latter become

$$\begin{aligned}
 (x) \quad 202 \cdot 72 &= 6 \cdot 5294 \delta x_1 + 5 \cdot 4577 \delta x_2 + 3 \cdot 4658 h_1 + 8 \cdot 1355 h_2 \\
 &\quad + 6 \cdot 0139 h_3 + 1 \cdot 1640 p_1 + 0 \cdot 8109 p_2 + 6 \cdot 0533 p_3 \\
 (h_1) \quad 111 \cdot 41 &= 3 \cdot 4658 \delta x_1 + 2 \cdot 6458 \delta x_2 + 1 \cdot 8531 h_1 + 4 \cdot 2731 h_2 \\
 &\quad + 2 \cdot 9588 h_3 + 0 \cdot 6243 p_1 + 0 \cdot 4349 p_2 + 3 \cdot 2183 p_3 \\
 (h_2) \quad 239 \cdot 76 &= 8 \cdot 1355 \delta x_1 + 7 \cdot 6504 \delta x_2 + 4 \cdot 2731 h_1 + 10 \cdot 2882 h_2 \\
 &\quad + 8 \cdot 2822 h_3 + 1 \cdot 4284 p_1 + 0 \cdot 9944 p_2 + 7 \cdot 5252 p_3 \\
 (h_3) \quad 119 \cdot 57 &= 6 \cdot 0139 \delta x_1 + 9 \cdot 5389 \delta x_2 + 2 \cdot 9588 h_1 + 8 \cdot 2822 h_2 \\
 &\quad + 9 \cdot 7166 h_3 + 0 \cdot 9593 p_1 + 0 \cdot 6652 p_2 + 5 \cdot 4866 p_3 \\
 (p_2) \quad 26 \cdot 50 &= 0 \cdot 8109 \delta x_1 + 0 \cdot 5900 \delta x_2 + 0 \cdot 4349 h_1 + 0 \cdot 9944 h_2 \\
 &\quad + 0 \cdot 6652 h_3 + 0 \cdot 1470 p_1 + 0 \cdot 1024 p_2 + 0 \cdot 7535 p_3 \\
 (p_3) \quad 189 \cdot 38 &= 6 \cdot 0533 \delta x_1 + 4 \cdot 9643 \delta x_2 + 3 \cdot 2183 h_1 + 7 \cdot 5252 h_2 \\
 &\quad + 5 \cdot 4866 h_3 + 1 \cdot 0815 p_1 + 0 \cdot 7535 p_2 + 5 \cdot 6139 p_3
 \end{aligned}$$

17. Again, by means of ( $x$ ) eliminate  $\delta x_i$  from each of the other equations, and we find

$$\begin{aligned}
 (h_1) \quad 3 \cdot 807 &= -0 \cdot 2512 \delta x_2 + 0 \cdot 0135 h_1 - 0 \cdot 0452 h_2 - 0 \cdot 2334 h_3 \\
 &\quad + 0 \cdot 0065 p_1 + 0 \cdot 0045 p_2 + 0 \cdot 0052 p_3 \\
 (h_2) \quad -12 \cdot 821 &= 0 \cdot 8502 \delta x_2 - 0 \cdot 0452 h_1 + 0 \cdot 1515 h_2 + 0 \cdot 7890 h_3 \\
 &\quad - 0 \cdot 0219 p_1 - 0 \cdot 0160 p_2 - 0 \cdot 0171 p_3 \\
 (h_3) \quad -67 \cdot 149 &= 4 \cdot 5120 \delta x_2 - 0 \cdot 2334 h_1 + 0 \cdot 7890 h_2 + 4 \cdot 1775 h_3 \\
 &\quad - 0 \cdot 1128 p_1 - 0 \cdot 0817 p_2 - 0 \cdot 0888 p_3 \\
 (p_2) \quad 1 \cdot 327 &= -0 \cdot 0878 \delta x_2 + 0 \cdot 0045 h_1 - 0 \cdot 0160 h_2 - 0 \cdot 0817 h_3 \\
 &\quad + 0 \cdot 0024 p_1 + 0 \cdot 0017 p_2 + 0 \cdot 0018 p_3 \\
 (p_3) \quad 1 \cdot 448 &= -0 \cdot 0955 \delta x_2 + 0 \cdot 0052 h_1 - 0 \cdot 0171 h_2 - 0 \cdot 0888 h_3 \\
 &\quad + 0 \cdot 0024 p_1 + 0 \cdot 0018 p_2 + 0 \cdot 0020 p_3
 \end{aligned}$$

18. Similarly, the equations of the second group are found to be

$$\begin{aligned}
 (n) \quad -171 \cdot 27 &= 77 \cdot 0000 \delta n + 9 \cdot 3938 \delta y_1 - 1 \cdot 2183 \delta y_2 \\
 &\quad + 8 \cdot 8463 h_1 + 7 \cdot 3034 h_2 - 0 \cdot 5927 h_3 \\
 &\quad + 5 \cdot 7519 q_1 + 4 \cdot 8755 q_2 + 9 \cdot 5583 q_3 \\
 (y) \quad -166 \cdot 33 &= 93 \cdot 9380 \delta n + 12 \cdot 7179 \delta y_1 + 1 \cdot 8907 \delta y_2 \\
 &\quad + 11 \cdot 2022 h_1 + 11 \cdot 0848 h_2 + 2 \cdot 6731 h_3 \\
 &\quad + 7 \cdot 0956 q_1 + 5 \cdot 9913 q_2 + 12 \cdot 7441 q_3
 \end{aligned}$$



$$\begin{aligned}
 (k_1) \quad -182 \cdot 87 &= 88 \cdot 4630 \delta n + 11 \cdot 2022 \delta y_1 - 0 \cdot 3210 \delta y_2 \\
 &\quad + 10 \cdot 2978 k_1 + 9 \cdot 0964 k_2 + 0 \cdot 4061 k_3 \\
 &\quad + 6 \cdot 6370 q_1 + 5 \cdot 6163 q_2 + 11 \cdot 3346 q_3 \\
 (k_2) \quad -89 \cdot 07 &= 73 \cdot 0340 \delta n + 11 \cdot 0848 \delta y_1 + 4 \cdot 8266 \delta y_2 \\
 &\quad + 9 \cdot 0964 k_1 + 10 \cdot 7040 k_2 + 5 \cdot 4376 k_3 \\
 &\quad + 5 \cdot 5855 q_1 + 4 \cdot 6976 q_2 + 10 \cdot 9375 q_3 \\
 (k_3) \quad +124 \cdot 80 &= -5 \cdot 9270 \delta n + 2 \cdot 6731 \delta y_1 + 10 \cdot 4253 \delta y_2 \\
 &\quad + 0 \cdot 4061 k_1 + 5 \cdot 4376 k_2 + 10 \cdot 2929 k_3 \\
 &\quad - 0 \cdot 2497 q_1 - 0 \cdot 2643 q_2 + 2 \cdot 1788 q_3 \\
 (q_2) \quad -107 \cdot 02 &= 48 \cdot 7550 \delta n + 5 \cdot 9913 \delta y_1 - 0 \cdot 6614 \delta y_2 \\
 &\quad + 5 \cdot 6163 k_1 + 4 \cdot 6976 k_2 - 0 \cdot 2643 k_3 \\
 &\quad + 3 \cdot 6475 q_1 + 3 \cdot 0894 q_2 + 6 \cdot 0897 q_3 \\
 (q_3) \quad -175 \cdot 89 &= 95 \cdot 5830 \delta n + 12 \cdot 7441 \delta y_1 + 1 \cdot 3845 \delta y_2 \\
 &\quad + 11 \cdot 3346 k_1 + 10 \cdot 9375 k_2 + 2 \cdot 1788 k_3 \\
 &\quad + 7 \cdot 2084 q_1 + 6 \cdot 0897 q_2 + 12 \cdot 7981 q_3
 \end{aligned}$$

19. By means of (n), eliminate  $\delta n$  from each of the other equations, and we have

$$\begin{aligned}
 (y) \quad 42 \cdot 61 &= 1 \cdot 2578 \delta y_1 + 3 \cdot 3771 \delta y_2 + 0 \cdot 4100 k_1 + 2 \cdot 1748 k_2 \\
 &\quad + 3 \cdot 3962 k_3 + 0 \cdot 0785 q_1 + 0 \cdot 0433 q_2 + 1 \cdot 0833 q_3 \\
 (k_1) \quad 13 \cdot 90 &= 0 \cdot 4100 \delta y_1 + 1 \cdot 0787 \delta y_2 + 0 \cdot 1346 k_1 + 0 \cdot 7057 k_2 \\
 &\quad + 1 \cdot 0871 k_3 + 0 \cdot 0288 q_1 + 0 \cdot 0150 q_2 + 0 \cdot 3534 q_3 \\
 (k_2) \quad 73 \cdot 38 &= 2 \cdot 1748 \delta y_1 + 5 \cdot 9822 \delta y_2 + 0 \cdot 7057 k_1 + 3 \cdot 7767 k_2 \\
 &\quad + 5 \cdot 9998 k_3 + 0 \cdot 1298 q_1 + 0 \cdot 0782 q_2 + 1 \cdot 8715 q_3 \\
 (k_3) \quad 111 \cdot 62 &= 3 \cdot 3962 \delta y_1 + 10 \cdot 3315 \delta y_2 + 1 \cdot 0871 k_1 + 5 \cdot 9998 k_2 \\
 &\quad + 10 \cdot 2473 k_3 + 0 \cdot 1930 q_1 + 0 \cdot 1110 q_2 + 2 \cdot 9145 q_3 \\
 (q_2) \quad 1 \cdot 42 &= 0 \cdot 0433 \delta y_1 + 0 \cdot 1100 \delta y_2 + 0 \cdot 0150 k_1 + 0 \cdot 0732 k_2 \\
 &\quad + 0 \cdot 1110 k_3 + 0 \cdot 0055 q_1 + 0 \cdot 0023 q_2 + 0 \cdot 0375 q_3 \\
 (q_3) \quad 36 \cdot 72 &= 1 \cdot 0833 \delta y_1 + 2 \cdot 8969 \delta y_2 + 0 \cdot 3534 k_1 + 1 \cdot 8715 k_2 \\
 &\quad + 2 \cdot 9145 k_3 + 0 \cdot 0684 q_1 + 0 \cdot 0375 q_2 + 0 \cdot 9330 q_3
 \end{aligned}$$

20. Again, eliminating  $\delta y_1$  by means of (y) we find

$$\begin{aligned}
 (k_1) \quad 0 \cdot 009 &= -0 \cdot 0221 \delta y_2 + 0 \cdot 0010 k_1 - 0 \cdot 0032 k_2 - 0 \cdot 0200 k_3 \\
 &\quad + 0 \cdot 0032 q_1 + 0 \cdot 0009 q_2 + 0 \cdot 0003 q_3 \\
 (k_2) \quad -0 \cdot 301 &= 0 \cdot 1430 \delta y_2 - 0 \cdot 0032 k_1 + 0 \cdot 0162 k_2 + 0 \cdot 1274 k_3 \\
 &\quad - 0 \cdot 0059 q_1 - 0 \cdot 0017 q_2 - 0 \cdot 0016 q_3 \\
 (k_3) \quad -3 \cdot 443 &= 1 \cdot 2129 \delta y_2 - 0 \cdot 0200 k_1 + 0 \cdot 1274 k_2 + 1 \cdot 0769 k_3 \\
 &\quad - 0 \cdot 0189 q_1 - 0 \cdot 0059 q_2 - 0 \cdot 0105 q_3 \\
 (q_2) \quad -0 \cdot 045 &= -0 \cdot 0062 \delta y_2 + 0 \cdot 0009 k_1 - 0 \cdot 0017 k_2 - 0 \cdot 0059 k_3 \\
 &\quad + 0 \cdot 0028 q_1 + 0 \cdot 0008 q_2 + 0 \cdot 0002 q_3 \\
 (q_3) \quad +0 \cdot 017 &= -0 \cdot 0116 \delta y_2 + 0 \cdot 0003 k_1 - 0 \cdot 0016 k_2 - 0 \cdot 0105 k_3 \\
 &\quad + 0 \cdot 0008 q_1 + 0 \cdot 0002 q_2 + 0 \cdot 0000 q_3
 \end{aligned}$$

21. From the equations remaining in the two groups after the elimination of  $\delta \epsilon$ ,  $\delta n$ ,  $\delta x_1$ ,  $\delta y_1$  it will be easy, when approximate values of the mass and mean longitude of the disturbing planet have been found, to deduce the final equations for determining these quantities more accurately by the method of minimum squares.

It may be observed, however, that the equations in each group are very nearly identical with each other, and therefore two final equations may be formed by simply adding together the several equations of each group, after giving the unknown quantities the same sign in them all. Thus we find

$$\begin{aligned} 86^{\prime\prime} \cdot 552 &= -5 \cdot 7967 \delta x_2 + 0 \cdot 3018 h_1 - 1 \cdot 0188 h_2 - 5 \cdot 3704 h_3 \\ &\quad + 0 \cdot 1460 p_1 + 0 \cdot 1056 p_2 + 0 \cdot 1149 p_3 \\ 3 \cdot 725 &= -1 \cdot 3958 \delta y_2 + 0 \cdot 0254 h_1 - 0 \cdot 1501 h_2 - 1 \cdot 2407 h_3 \\ &\quad + 0 \cdot 0316 q_1 + 0 \cdot 0095 q_2 + 0 \cdot 0127 q_3 \end{aligned}$$

22. If in the expressions before given for  $\delta x_2$  and  $\delta y_2$  we substitute  $e = 0 \cdot 046679$  and  $\epsilon - \omega = 50^\circ 15' \cdot 8$ , we obtain

$$\begin{aligned} \delta x_2 &= 0 \cdot 007460 \delta x_1 + 0 \cdot 008974 \delta y_1 \\ \delta y_2 &= -0 \cdot 008974 \delta x_1 + 0 \cdot 007460 \delta y_1 \end{aligned}$$

Substituting these values in the equations (x) and (y) and in those just found, it may be seen that by adding to the latter equations

$$0 \cdot 006768 (x) + 0 \cdot 040287 (y)$$

and

$$-0 \cdot 001869 (x) + 0 \cdot 008187 (y) \text{ respectively,}$$

$\delta x_1$  and  $\delta y_1$  will be eliminated, and we shall obtain the following equations :

$$\begin{aligned} (1) \quad 89^{\prime\prime} \cdot 641 &= 0 \cdot 3252 h_1 - 0 \cdot 9637 h_2 - 5 \cdot 3297 h_3 \\ &\quad + 0 \cdot 0165 k_1 + 0 \cdot 0876 k_2 + 0 \cdot 1368 k_3 \\ &\quad + 0 \cdot 1539 p_1 + 0 \cdot 1111 p_2 + 0 \cdot 1559 p_3 \\ &\quad + 0 \cdot 0032 q_1 + 0 \cdot 0017 q_2 + 0 \cdot 0436 q_3 \\ (2) \quad 3 \cdot 695 &= -0 \cdot 0065 h_1 - 0 \cdot 0152 h_2 - 0 \cdot 0112 h_3 \\ &\quad + 0 \cdot 0288 k_1 - 0 \cdot 1323 k_2 - 1 \cdot 2129 k_3 \\ &\quad - 0 \cdot 0022 p_1 - 0 \cdot 0015 p_2 - 0 \cdot 0113 p_3 \\ &\quad + 0 \cdot 0323 q_1 + 0 \cdot 0099 q_2 + 0 \cdot 0215 q_3 \end{aligned}$$

23. These equations would be sufficient for determining the mass of the disturbing planet and its longitude at the epoch, if the eccentricity of the orbit were neglected. We will now proceed to find equations from the Ancient Observations for determining the eccentricity and longitude of the Perihelion.

The equations of condition given by the Ancient Observations are the following :

$$\begin{aligned} 62^{\prime\prime} \cdot 6 &= \delta \epsilon - 0 \cdot 8776 \delta x_1 + 0 \cdot 5402 \delta x_2 + 0 \cdot 8712 h_1 + 0 \cdot 5180 h_2 \\ &\quad - 39 \cdot 31 \delta n - 0 \cdot 1795 \delta y_1 + 0 \cdot 8415 \delta y_2 + 0 \cdot 4909 k_1 + 0 \cdot 8554 k_2 \\ &\quad + 0 \cdot 0314 h_3 - 0 \cdot 9999 p_1 - 0 \cdot 8640 p_2 - 0 \cdot 5055 p_3 \\ &\quad + 0 \cdot 9995 k_3 + 0 \cdot 0145 q_1 - 0 \cdot 5035 q_2 - 0 \cdot 8628 q_3 \end{aligned}$$

$$\begin{aligned}
84.5 &= \delta\epsilon + 0.4975 \delta x_1 - 0.5050 \delta x_2 + 0.0288 h_1 - 0.9984 h_2 \\
&\quad - 32.30 \delta n - 0.8675 \delta y_1 - 0.8631 \delta y_2 + 0.9596 k_1 + 0.0573 k_2 \\
&\quad - 0.0860 h_3 - 0.8534 p_1 - 0.5456 p_2 + 0.8220 p_3 \\
&\quad - 0.9963 k_3 - 0.5213 q_1 - 0.8380 q_2 - 0.5695 q_3 \\
67.2 &= \delta\epsilon + 0.6732 \delta x_1 - 0.0935 \delta x_2 - 0.1120 h_1 - 0.9749 h_2 \\
&\quad - 31.34 \delta n - 0.7394 \delta y_1 - 0.9956 \delta y_2 + 0.9937 k_1 - 0.2227 k_2 \\
&\quad + 0.3305 h_3 - 0.8105 p_1 - 0.4912 p_2 + 0.9206 p_3 \\
&\quad - 0.9139 k_3 - 0.5857 q_1 - 0.8711 q_2 - 0.3905 q_3 \\
-51.8 &= \delta\epsilon - 0.2616 \delta x_1 - 0.8631 \delta x_2 - 0.9649 h_1 + 0.8618 h_2 \\
&\quad - 19.59 \delta n + 0.9652 \delta y_1 - 0.5050 \delta y_2 - 0.2627 k_1 + 0.5073 k_2 \\
&\quad - 0.6982 h_3 - 0.0023 p_1 + 0.2650 p_2 - 0.5090 p_3 \\
&\quad - 0.7159 k_3 - 1.0000 q_1 - 0.9642 q_2 + 0.8607 q_3 \\
-43.2 &= \delta\epsilon - 0.4741 \delta x_1 - 0.5505 \delta x_2 - 0.9154 h_1 + 0.6758 h_2 \\
&\quad - 18.58 \delta n + 0.8805 \delta y_1 - 0.8348 \delta y_2 - 0.4025 k_1 + 0.7371 k_2 \\
&\quad - 0.3220 h_3 + 0.0787 p_1 + 0.3291 p_2 - 0.6814 p_3 \\
&\quad - 0.9467 k_3 - 0.9969 q_1 - 0.9443 q_2 + 0.7319 q_3 \\
-50.1 &= \delta\epsilon - 0.6430 \delta x_1 - 0.1731 \delta x_2 - 0.8543 h_1 + 0.4599 h_2 \\
&\quad - 17.68 \delta n + 0.7659 \delta y_1 - 0.9849 \delta y_2 - 0.5198 k_1 + 0.8879 k_2 \\
&\quad + 0.0686 h_3 + 0.1510 p_1 + 0.3848 p_2 - 0.8085 p_3 \\
&\quad - 0.9976 k_3 - 0.9885 q_1 - 0.9230 q_2 + 0.5885 q_3 \\
-37.8 &= \delta\epsilon - 0.9492 \delta x_1 + 0.8021 \delta x_2 - 0.6189 h_1 - 0.2340 h_2 \\
&\quad - 15.25 \delta n + 0.3145 \delta y_1 - 0.5972 \delta y_2 - 0.7855 k_1 + 0.9722 k_2 \\
&\quad + 0.9085 h_3 + 0.3396 p_1 + 0.5287 p_2 - 0.9939 p_3 \\
&\quad - 0.4179 k_3 - 0.9406 q_1 - 0.8488 q_2 + 0.1100 q_3 \\
-20.5 &= \delta\epsilon - 0.9985 \delta x_1 + 0.9942 \delta x_2 - 0.4128 h_1 - 0.6591 h_2 \\
&\quad - 13.60 \delta n - 0.0538 \delta y_1 + 0.1074 \delta y_2 - 0.9108 k_1 + 0.7520 k_2 \\
&\quad + 0.9571 h_3 + 0.4607 p_1 + 0.6182 p_2 - 0.9711 p_3 \\
&\quad + 0.2899 k_3 - 0.8875 q_1 - 0.7860 q_2 - 0.2385 q_3 \\
-2.4 &= \delta\epsilon - 0.9633 \delta x_1 + 0.8560 \delta x_2 - 0.2807 h_1 - 0.8424 h_2 \\
&\quad - 12.64 \delta n - 0.2684 \delta y_1 + 0.5170 \delta y_2 - 0.9598 k_1 + 0.5388 k_2 \\
&\quad + 0.7536 h_3 + 0.5279 p_1 + 0.6670 p_2 - 0.9023 p_3 \\
&\quad + 0.6574 k_3 - 0.8493 q_1 - 0.7451 q_2 - 0.4310 q_3
\end{aligned}$$

24. From each of these equations eliminate  $\delta\epsilon$ ,  $\delta n$ ,  $\delta x_1$ , and  $\delta y_1$ , by means of the equations ( $\epsilon$ ), ( $n$ ), ( $x$ ), and ( $y$ ) before found, and we have the following :

$$\begin{aligned}
-142.0 &= 1.7265 \delta x_2 + 0.8412 h_1 + 1.9521 h_2 + 1.3230 h_3 \\
&\quad - 11.3691 \delta y_2 + 3.6001 k_1 - 2.8793 k_2 - 10.9578 k_3 \\
&\quad - 1.6779 p_1 - 1.6400 p_2 + 0.2249 p_3 \\
&\quad + 2.6815 q_1 + 1.8369 q_2 + 0.2995 q_3
\end{aligned}$$

$$\begin{aligned}
-105 \cdot 2 &= -0 \cdot 4681 \delta x_2 - 0 \cdot 7311 h_1 - 1 \cdot 2776 h_2 - 0 \cdot 0609 h_3 \\
&\quad - 9 \cdot 6249 \delta y_2 + 3 \cdot 7087 k_1 - 2 \cdot 1926 k_2 - 9 \cdot 5426 k_3 \\
&\quad \quad - 1 \cdot 7765 p_1 - 1 \cdot 4924 p_2 + 0 \cdot 2786 p_3 \\
&\quad \quad + 1 \cdot 6997 q_1 + 1 \cdot 1014 q_2 + 0 \cdot 7934 q_3 \\
-126 \cdot 1 &= -0 \cdot 2035 \delta x_2 - 0 \cdot 9653 h_1 - 1 \cdot 4730 h_2 + 0 \cdot 1937 h_3 \\
&\quad - 9 \cdot 7719 \delta y_2 + 3 \cdot 5895 k_1 - 2 \cdot 5827 k_2 - 9 \cdot 5123 k_3 \\
&\quad \quad - 1 \cdot 7649 p_1 - 1 \cdot 4598 p_2 + 0 \cdot 2133 p_3 \\
&\quad \quad + 1 \cdot 5629 q_1 + 1 \cdot 0070 q_2 + 0 \cdot 8437 q_3 \\
-199 \cdot 1 &= -0 \cdot 1917 \delta x_2 - 1 \cdot 3218 h_1 + 1 \cdot 5284 h_2 + 0 \cdot 0260 h_3 \\
&\quad - 9 \cdot 8232 \delta y_2 + 0 \cdot 8943 k_1 - 3 \cdot 4359 k_2 - 9 \cdot 9270 k_3 \\
&\quad \quad - 0 \cdot 7901 p_1 - 0 \cdot 5885 p_2 - 0 \cdot 3497 p_3 \\
&\quad \quad + 0 \cdot 2540 q_1 + 0 \cdot 1607 q_2 + 0 \cdot 4028 q_3 \\
-174 \cdot 7 &= 0 \cdot 2985 \delta x_2 - 1 \cdot 1595 h_1 + 1 \cdot 6072 h_2 + 0 \cdot 5979 h_3 \\
&\quad - 9 \cdot 5788 \delta y_2 + 0 \cdot 7062 k_1 - 2 \cdot 9425 k_2 - 9 \cdot 5877 k_3 \\
&\quad \quad - 0 \cdot 6712 p_1 - 0 \cdot 4970 p_2 - 0 \cdot 3251 p_3 \\
&\quad \quad + 0 \cdot 1946 q_1 + 0 \cdot 1238 q_2 + 0 \cdot 3277 q_3 \\
-166 \cdot 7 &= 0 \cdot 8171 \delta x_2 - 1 \cdot 0088 h_1 + 1 \cdot 6018 h_2 + 1 \cdot 1442 h_3 \\
&\quad - 9 \cdot 1122 \delta y_2 + 0 \cdot 5586 k_1 - 2 \cdot 4890 k_2 - 9 \cdot 0258 k_3 \\
&\quad \quad - 0 \cdot 5688 p_1 - 0 \cdot 4203 p_2 - 0 \cdot 2956 p_3 \\
&\quad \quad + 0 \cdot 1498 q_1 + 0 \cdot 0958 q_2 + 0 \cdot 2658 q_3 \\
-114 \cdot 2 &= 2 \cdot 0482 \delta x_2 - 0 \cdot 6027 h_1 + 1 \cdot 2894 h_2 + 2 \cdot 2661 h_3 \\
&\quad - 6 \cdot 6781 \delta y_2 + 0 \cdot 2576 k_1 - 1 \cdot 3421 k_2 - 6 \cdot 4080 k_3 \\
&\quad \quad - 0 \cdot 3256 p_1 - 0 \cdot 2384 p_2 - 0 \cdot 1971 p_3 \\
&\quad \quad + 0 \cdot 0628 q_1 + 0 \cdot 0419 q_2 + 0 \cdot 1298 q_3 \\
-72 \cdot 4 &= 2 \cdot 2815 \delta x_2 - 0 \cdot 3786 h_1 + 0 \cdot 9257 h_2 + 2 \cdot 3601 h_3 \\
&\quad - 4 \cdot 4181 \delta y_2 + 0 \cdot 1283 k_1 - 0 \cdot 7339 k_2 - 4 \cdot 1495 k_3 \\
&\quad \quad - 0 \cdot 1957 p_1 - 0 \cdot 1428 p_2 - 0 \cdot 1286 p_3 \\
&\quad \quad + 0 \cdot 0283 q_1 + 0 \cdot 0198 q_2 + 0 \cdot 0671 q_3 \\
-42 \cdot 0 &= 2 \cdot 1139 \delta x_2 - 0 \cdot 2652 h_1 + 0 \cdot 6985 h_2 + 2 \cdot 1241 h_3 \\
&\quad - 3 \cdot 1027 \delta y_2 + 0 \cdot 0772 k_1 - 0 \cdot 4616 k_2 - 2 \cdot 8790 k_3 \\
&\quad \quad - 0 \cdot 1348 p_1 - 0 \cdot 0984 p_2 - 0 \cdot 0924 p_3 \\
&\quad \quad + 0 \cdot 0154 q_1 + 0 \cdot 0114 q_2 + 0 \cdot 0412 q_3
\end{aligned}$$

25. The largest terms depending on the eccentricity of the disturbing planet occur in  $p_2, q_2$ ; it will be proper, therefore, to combine the above equations in such a manner that these quantities may acquire the largest co-efficients possible. This will be done by multiplying each equation by a quantity nearly proportional to the co-efficient of each of the unknown quantities  $p_2$  and  $q_2$ , and adding together the several results. It was thought unsafe to employ the first of the above equations, since it is derived from the single observation of Flamsteed, made in 1690, twenty-two years anterior to any other observation.

Hence the equation for finding  $p_1$  may be formed by multiplying the above equations, taken in order, by

$$-0 \cdot 3, -0 \cdot 6, +1 \cdot 0, +1 \cdot 0, +0 \cdot 9, +0 \cdot 6, +0 \cdot 4, +0 \cdot 3$$

beginning with the second; and the equation for  $q_1$  by multiplying the same equations by

$$1 \cdot 0, 1 \cdot 0, 0 \cdot 5, 0 \cdot 4, 0 \cdot 3, 0 \cdot 2, 0 \cdot 1, 0 \cdot 1,$$

Hence we obtain

$$\begin{aligned} -474 \cdot 1 = & 4 \cdot 114 \delta x_2 - 2 \cdot 817 h_1 + 7 \cdot 837 h_2 + 4 \cdot 528 h_3 \\ & - 20 \cdot 745 \delta y_2 - 2 \cdot 789 k_1 - 6 \cdot 551 k_2 - 20 \cdot 666 k_3 \\ & + 0 \cdot 193 p_1 + 0 \cdot 377 p_2 - 1 \cdot 489 p_3 \\ & - 1 \cdot 660 q_1 - 1 \cdot 078 q_2 - 0 \cdot 054 q_3 \end{aligned}$$

$$\begin{aligned} -485 \cdot 0 = & 0 \cdot 446 \delta x_2 - 3 \cdot 308 h_1 - 0 \cdot 442 h_2 + 1 \cdot 629 h_3 \\ & - 32 \cdot 961 \delta y_2 + 8 \cdot 267 k_1 - 8 \cdot 805 k_2 - 32 \cdot 546 k_3 \\ & - 4 \cdot 473 p_1 - 3 \cdot 643 p_2 + 0 \cdot 037 p_3 \\ & + 3 \cdot 530 q_1 + 2 \cdot 278 q_2 + 2 \cdot 086 q_3 \end{aligned}$$

26. Eliminate  $\delta x_2$  and  $\delta y_2$  from these equations by means of (x) and (y) and they become

$$\begin{aligned} (3) \quad -476 \cdot 7 = & -2 \cdot 930 h_1 + 7 \cdot 572 h_2 + 4 \cdot 332 h_3 \\ & - 2 \cdot 751 k_1 - 6 \cdot 348 k_2 - 20 \cdot 350 k_3 \\ & + 0 \cdot 155 p_1 + 0 \cdot 350 p_2 - 1 \cdot 686 p_3 \\ & - 1 \cdot 653 q_1 - 1 \cdot 074 q_2 + 0 \cdot 047 q_3 \end{aligned}$$

$$\begin{aligned} (4) \quad -485 \cdot 9 = & -3 \cdot 463 h_1 - 0 \cdot 805 h_2 + 1 \cdot 360 h_3 \\ & + 8 \cdot 345 k_1 - 8 \cdot 391 k_2 - 31 \cdot 900 k_3 \\ & - 4 \cdot 525 p_1 - 3 \cdot 679 p_2 - 0 \cdot 233 p_3 \\ & + 3 \cdot 545 q_1 + 2 \cdot 286 q_2 + 2 \cdot 292 q_3 \end{aligned}$$

These equations, with (1) and (2) of Article 22, suffice for the solution of our problem.

27. Eliminate the left hand members from equations (2), (3), (4), by means of equation (1) and we have

$$\begin{aligned} 0 = & 0 \cdot 4819 h_1 - 0 \cdot 5950 h_2 - 5 \cdot 0570 h_3 + 0 \cdot 2063 p_1 + 0 \cdot 1475 p_2 + 0 \cdot 4300 p_3 \\ & - 0 \cdot 6812 k_1 + 3 \cdot 2982 k_2 + 29 \cdot 5618 k_3 - 0 \cdot 7804 q_1 - 0 \cdot 2375 q_2 - 0 \cdot 4789 q_3 \end{aligned}$$

$$\begin{aligned} 0 = & -1 \cdot 2005 h_1 + 2 \cdot 4466 h_2 - 24 \cdot 0122 h_3 + 0 \cdot 9735 p_1 + 0 \cdot 9412 p_2 - 0 \cdot 8575 p_3 \\ & - 2 \cdot 6633 k_1 - 5 \cdot 8825 k_2 - 19 \cdot 6219 k_3 - 1 \cdot 6362 q_1 - 1 \cdot 0648 q_2 + 0 \cdot 2791 q_3 \end{aligned}$$

$$\begin{aligned} 0 = & -1 \cdot 7003 h_1 - 6 \cdot 0294 h_2 - 27 \cdot 5295 h_3 - 3 \cdot 6908 p_1 - 3 \cdot 0772 p_2 + 0 \cdot 6118 p_3 \\ & + 8 \cdot 4344 k_1 - 7 \cdot 9162 k_2 - 31 \cdot 1583 k_3 + 3 \cdot 5621 q_1 + 2 \cdot 2954 q_2 + 2 \cdot 5285 q_3 \end{aligned}$$

28. If now we put  $\epsilon - \epsilon' = \theta$  and  $\epsilon - \omega = \beta$ , it is easily seen that

$$\begin{aligned}\frac{h_1}{m'} &= -36 \cdot 99 \sin \theta, & \frac{h_2}{m'} &= 58 \cdot 97 \sin 2 \theta \\ \frac{k_1}{m'} &= -36 \cdot 99 \cos \theta, & \frac{k_2}{m'} &= 58 \cdot 97 \cos 2 \theta \\ \frac{h_3}{m'} &= 5 \cdot 80 \sin 3 \theta & + 0 \cdot 007460 \frac{p_3}{m'} + 0 \cdot 008974 \frac{q_3}{m'} \\ \frac{k_3}{m'} &= 5 \cdot 80 \cos 3 \theta & - 0 \cdot 008974 \frac{p_3}{m'} + 0 \cdot 007460 \frac{q_3}{m'} \\ \frac{p_1}{m'} &= 0 \cdot 18 \sin (\theta - \beta) & - 0 \cdot 046247 \left\{ \frac{p_3}{m'} \cos 2 \theta - \frac{q_3}{m'} \sin 2 \theta \right\} \\ \frac{q_1}{m'} &= -0 \cdot 18 \cos (\theta - \beta) & + 0 \cdot 046247 \left\{ \frac{p_3}{m'} \sin 2 \theta + \frac{q_3}{m'} \cos 2 \theta \right\} \\ \frac{p_2}{m'} &= 24 \cdot 91 \sin (2 \theta - \beta) & + 0 \cdot 13055 \left\{ \frac{p_3}{m'} \cos \theta - \frac{q_3}{m'} \sin \theta \right\} \\ \frac{q_2}{m'} &= 24 \cdot 91 \cos (2 \theta - \beta) & + 0 \cdot 13055 \left\{ \frac{p_3}{m'} \sin \theta + \frac{q_3}{m'} \cos \theta \right\}\end{aligned}$$

29. Substituting these expressions in the equations of Art. 27, and putting for  $\beta$  its value  $50^\circ 15' \cdot 8$ , we obtain, after a slight reduction,

$$\begin{aligned}0 &= -(1 \cdot 21782) \sin \theta + (1 \cdot 40248) \cos \theta - (1 \cdot 57155) \sin 2 \theta + (2 \cdot 27388) \cos 2 \theta \\ &\quad - (1 \cdot 46746) \sin 3 \theta + (2 \cdot 23130) \cos 3 \theta + (9 \cdot 10380) \frac{p_3}{m'} - (9 \cdot 48251) \frac{q_3}{m'} \\ &\quad + (8 \cdot 28455) \left\{ \frac{p_3}{m'} \cos \theta - \frac{q_3}{m'} \sin \theta \right\} - (8 \cdot 49138) \left\{ \frac{p_3}{m'} \sin \theta + \frac{q_3}{m'} \cos \theta \right\} \\ &\quad - (7 \cdot 97958) \left\{ \frac{p_3}{m'} \cos 2 \theta - \frac{q_3}{m'} \sin 2 \theta \right\} - (8 \cdot 55742) \left\{ \frac{p_3}{m'} \sin 2 \theta + \frac{q_3}{m'} \cos 2 \theta \right\} \\ 0 &= (1 \cdot 65083) \sin \theta + (1 \cdot 99378) \cos \theta + (2 \cdot 14259) \sin 2 \theta - (2 \cdot 58192) \cos 2 \theta \\ &\quad - (2 \cdot 14400) \sin 3 \theta - (2 \cdot 05631) \cos 3 \theta - (9 \cdot 93475) \frac{p_3}{m'} - (8 \cdot 91803) \frac{q_3}{m'} \\ &\quad + (9 \cdot 08947) \left\{ \frac{p_3}{m'} \cos \theta - \frac{q_3}{m'} \sin \theta \right\} - (9 \cdot 14306) \left\{ \frac{p_3}{m'} \sin \theta + \frac{q_3}{m'} \cos \theta \right\} \\ &\quad - (8 \cdot 65341) \left\{ \frac{p_3}{m'} \cos 2 \theta - \frac{q_3}{m'} \sin 2 \theta \right\} - (8 \cdot 87892) \left\{ \frac{p_3}{m'} \sin 2 \theta + \frac{q_3}{m'} \cos 2 \theta \right\} \\ 0 &= (1 \cdot 79213) \sin \theta - (2 \cdot 49403) \cos \theta - (2 \cdot 55700) \sin 2 \theta - (2 \cdot 56972) \cos 2 \theta \\ &\quad - (2 \cdot 20337) \sin 3 \theta - (2 \cdot 25714) \cos 3 \theta + (9 \cdot 83632) \frac{p_3}{m'} + (0 \cdot 31156) \frac{q_3}{m'} \\ &\quad - (9 \cdot 60395) \left\{ \frac{p_3}{m'} \cos \theta - \frac{q_3}{m'} \sin \theta \right\} + (9 \cdot 47665) \left\{ \frac{p_3}{m'} \sin \theta + \frac{q_3}{m'} \cos \theta \right\} \\ &\quad + (9 \cdot 23220) \left\{ \frac{p_3}{m'} \cos 2 \theta - \frac{q_3}{m'} \sin 2 \theta \right\} + (9 \cdot 21679) \left\{ \frac{p_3}{m'} \sin 2 \theta + \frac{q_3}{m'} \cos 2 \theta \right\}\end{aligned}$$

where the numbers enclosed within parentheses denote the logarithms of the corresponding coefficients.

30. These equations may be rapidly solved by approximation. The coefficients of  $\frac{p_s}{m'}$  and  $\frac{q_s}{m'}$  in the first equation being small, we may find from it an approximate value of  $\theta$ , the substitution of which in the second and third equations will give approximate values of  $\frac{p_s}{m'}$  and  $\frac{q_s}{m'}$ . By means of these a more accurate value of  $\theta$  may be found from the first equation, and the process being repeated will enable us to satisfy all the equations as nearly as we please.

$$\text{Thus we find } \theta = -51^\circ 30', \quad \frac{p_s}{m'} = 271'' \cdot 57, \quad \frac{q_s}{m'} = -207'' \cdot 24.$$

Now  $\epsilon$  is known and  $=217^\circ 55'$ .  $\therefore \epsilon' = 269^\circ 25'$  the mean longitude of the disturbing planet at the epoch 1810·328. The sidereal motion in 36 synodic periods of URANUS  $= 55^\circ 12'$ , Precession  $= 30'$ ,  $\therefore$  Mean Longitude at the time 1846·762, or October 6, 1846,  $= 325^\circ 7'$ .

Also, the analytical expressions for  $\frac{p_s}{m'}$  and  $\frac{q_s}{m'}$  are

$$\frac{p_s}{m'} = 48'' \cdot 55 \sin(3\theta - \beta) - 93'' \cdot 01 \epsilon' \sin(3\theta - \beta')$$

$$\frac{q_s}{m'} = 48'' \cdot 55 \cos(3\theta - \beta) - 93'' \cdot 01 \epsilon' \cos(3\theta - \beta')$$

where  $\epsilon - \omega' = \beta'$ . Equating these to the values given above, we find  $\epsilon' = 3^\circ 22'06$ ,  $\beta' = 262^\circ 28'$ , and  $\therefore \omega' = 315^\circ 27'$ . Hence long. of Perihelion in 1846  $= 315^\circ 57'$ .

Lastly, substituting the values just obtained in equation (1), we find  $m' = 0 \cdot 82816$ .

31. Hence the values of the mass and elements of the orbit of the disturbing planet, resulting from the first hypothesis as to the mean distance, are the following:

$$\frac{a}{a'} = 0 \cdot 5$$

Mean Long. of the planet, October 6, 1846,	325	7
Longitude of the Perihelion - - - - -	315	57
Eccentricity of the Orbit - - - - -	0	16103
Mass (that of the SUN being 1) - - - - -	0	0001656

These are the results which I communicated to the Astronomer Royal in October, 1845.

32. I next entered upon a similar investigation, founded on the assumption that the mean distance was about  $\frac{1}{8}$ th part less than before, so that  $\frac{a}{a'}$  or  $\alpha = \sin 31^\circ = 0 \cdot 515$ . The method employed was, in principle, exactly the same as that given before; but the numerical calculations were somewhat shortened by a few alterations in the process, which had been suggested by my previous solution.

33. Assuming then that  $\alpha = \sin 31^\circ$ , the values of the quantities  $b$ ,  $\alpha \frac{db}{d\alpha}$ ,  $\alpha^2 \frac{d^2b}{d\alpha^2}$  will be

$$\begin{aligned} \log. b_0 &= 0.33385; & \log. \alpha \frac{db_0}{d\alpha} &= 9.57333; & \log. \alpha^2 \frac{d^2b_0}{d\alpha^2} &= 9.82911 \\ \log. b_1 &= 9.76106; & \log. \alpha \frac{db_1}{d\alpha} &= 9.86149; & \log. \alpha^2 \frac{d^2b_1}{d\alpha^2} &= 9.76573 \\ \log. b_2 &= 9.35361; & \log. \alpha \frac{db_2}{d\alpha} &= 9.71359; & \log. \alpha^2 \frac{d^2b_2}{d\alpha^2} &= 9.92466 \\ \log. b_3 &= 8.98918; & \log. \alpha \frac{db_3}{d\alpha} &= 9.50854; & \log. \alpha^2 \frac{d^2b_3}{d\alpha^2} &= 9.91563 \end{aligned}$$

Hence, by means of the formulæ given before, the principal inequalities of the mean longitude of URANUS, produced by the action of a planet whose mass is  $\frac{m'}{5000}$ , that of the SUN being unity, and the eccentricity of whose orbit is  $\frac{e'}{20}$ , may be found to be the following :

$$\begin{aligned} & -42.33 m' \sin \{nt - n't + \varepsilon - \varepsilon'\} \\ & + 76.55 m' \sin 2 \{nt - n't + \varepsilon - \varepsilon'\} \\ & + 7.25 m' \sin 3 \{nt - n't + \varepsilon - \varepsilon'\} \\ & + 2.34 m' \sin \{n't + \varepsilon' - \omega\} \\ & - 4.74 m' e' \sin \{n't + \varepsilon' - \omega'\} \\ & + 41.72 m' \sin \{nt - 2n't + \varepsilon - 2\varepsilon' + \omega\} \\ & - 16.47 m' e' \sin \{nt - 2n't + \varepsilon - 2\varepsilon' + \omega'\} \\ & + 33.93 m' \sin \{2nt - 3n't + 2\varepsilon - 3\varepsilon' + \omega\} \\ & - 63.41 m' e' \sin \{2nt - 3n't + 2\varepsilon - 3\varepsilon' + \omega'\} \end{aligned}$$

To these we may add the following, which are of two dimensions in terms of the eccentricities :

$$\begin{aligned} & + 0.40 m' \sin 3 \{nt - n't + \varepsilon - \varepsilon'\} \\ & - 0.74 m' e' \sin \{3 (nt - n't + \varepsilon - \varepsilon') - \omega + \omega'\} \end{aligned}$$

34. Now, on our present assumption,  $n = 13^\circ 0' .6$ ,  $n' = 4^\circ 48' .5$ ,  $n - n' = 8^\circ 12' .1$ ,  $n - 2n' = 3^\circ 23' .6$ ,  $2n - 3n' = 11^\circ 35' .7$ .

Hence the equations of condition given by the modern observations will be of the form

$$\begin{aligned} c = & \delta\varepsilon + \delta x_1 \cos \{13 \ 0 \ .5\} t + \delta x_2 \cos \{26 \ 1 \ .0\} t \\ & + t \delta n + \delta y_1 \sin \{13 \ 0 \ .5\} t + \delta y_2 \sin \{26 \ 1 \ .0\} t \\ & + h_1 \cos \{8 \ 12 \ .1\} t + h_2 \cos \{16 \ 24 \ .2\} t + h_3 \cos \{24 \ 36 \ .3\} t \\ & + k_1 \sin \{8 \ 12 \ .1\} t + k_2 \sin \{16 \ 24 \ .2\} t + k_3 \sin \{24 \ 36 \ .3\} t \\ & + p_1 \cos \{4 \ 48 \ .5\} t + p_2 \cos \{3 \ 23 \ .6\} t + p_3 \cos \{11 \ 35 \ .7\} t \\ & + q_1 \sin \{4 \ 48 \ .5\} t + q_2 \sin \{3 \ 23 \ .6\} t + q_3 \sin \{11 \ 35 \ .7\} t \end{aligned}$$



35. Treating these equations of condition in the same manner as before, the equations in the first group, derived from them, are found to be the following :

$$\begin{aligned}
 (\epsilon) \quad 151 \cdot 48 &= 21 \cdot 0000 \delta\epsilon + 6 \cdot 0670 \delta x_1 - 4 \cdot 4358 \delta x_2 \\
 &\quad + 13 \cdot 9515 h_1 + 0 \cdot 9471 h_2 - 4 \cdot 5965 h_3 \\
 &\quad + 18 \cdot 3916 p_1 + 19 \cdot 6752 p_2 + 8 \cdot 4184 p_3 \\
 (x) \quad 216 \cdot 48 &= 6 \cdot 0670 \delta\epsilon + 8 \cdot 2821 \delta x_1 + 4 \cdot 1762 \delta x_2 \\
 &\quad + 7 \cdot 3540 h_1 + 8 \cdot 3027 h_2 + 5 \cdot 0961 h_3 \\
 &\quad + 6 \cdot 5793 p_1 + 6 \cdot 3319 p_2 + 8 \cdot 0850 p_3 \\
 (h_1) \quad 207 \cdot 58 &= 13 \cdot 9515 \delta\epsilon + 7 \cdot 3540 \delta x_1 - 0 \cdot 4177 \delta x_2 \\
 &\quad + 10 \cdot 9735 h_1 + 4 \cdot 6775 h_2 - 0 \cdot 0005 h_3 \\
 &\quad + 12 \cdot 8697 p_1 + 13 \cdot 4050 p_2 + 8 \cdot 4781 p_3 \\
 (h_2) \quad 245 \cdot 17 &= 0 \cdot 9471 \delta\epsilon + 8 \cdot 3027 \delta x_1 + 7 \cdot 2362 \delta x_2 \\
 &\quad + 4 \cdot 6775 h_1 + 10 \cdot 0259 h_2 + 8 \cdot 3220 h_3 \\
 &\quad + 2 \cdot 3661 p_1 + 1 \cdot 6727 p_2 + 7 \cdot 3073 p_3 \\
 (h_3) \quad 103 \cdot 48 &= -4 \cdot 5965 \delta\epsilon + 5 \cdot 0961 \delta x_1 + 10 \cdot 5558 \delta x_2 \\
 &\quad - 0 \cdot 0005 h_1 + 8 \cdot 3220 h_2 + 10 \cdot 9749 h_3 \\
 &\quad - 2 \cdot 8935 p_1 - 3 \cdot 7316 p_2 + 3 \cdot 5852 p_3
 \end{aligned}$$

36. Similarly the equations in the second group, are

$$\begin{aligned}
 (n) \quad -171 \cdot 27 &= 77 \cdot 0000 \delta n + 9 \cdot 3938 \delta y_1 - 1 \cdot 2183 \delta y_2 \\
 &\quad + 8 \cdot 7355 h_1 + 7 \cdot 6213 h_2 - 0 \cdot 0590 h_3 \\
 &\quad + 5 \cdot 9764 q_1 + 4 \cdot 3875 q_2 + 9 \cdot 6152 q_3 \\
 (y) \quad -166 \cdot 33 &= 93 \cdot 9380 \delta n + 12 \cdot 7179 \delta y_1 + 1 \cdot 8907 \delta y_2 \\
 &\quad + 11 \cdot 0393 h_1 + 11 \cdot 3717 h_2 + 3 \cdot 3196 h_3 \\
 &\quad + 7 \cdot 3747 q_1 + 5 \cdot 3825 q_2 + 12 \cdot 6816 q_3 \\
 (k_1) \quad -181 \cdot 31 &= 87 \cdot 3550 \delta n + 11 \cdot 0393 \delta y_1 - 0 \cdot 3758 \delta y_2 \\
 &\quad + 10 \cdot 0264 h_1 + 9 \cdot 2740 h_2 + 0 \cdot 9476 h_3 \\
 &\quad + 6 \cdot 8054 q_1 + 4 \cdot 9866 q_2 + 11 \cdot 1971 q_3 \\
 (k_2) \quad -99 \cdot 51 &= 76 \cdot 2130 \delta n + 11 \cdot 3717 \delta y_1 + 4 \cdot 4810 \delta y_2 \\
 &\quad + 9 \cdot 2740 h_1 + 10 \cdot 9740 h_2 + 5 \cdot 6294 h_3 \\
 &\quad + 6 \cdot 0523 q_1 + 4 \cdot 3916 q_2 + 11 \cdot 0843 q_3 \\
 (k_3) \quad 113 \cdot 14 &= -0 \cdot 5900 \delta n + 3 \cdot 3196 \delta y_1 + 10 \cdot 2112 \delta y_2 \\
 &\quad + 0 \cdot 9476 h_1 + 5 \cdot 6294 h_2 + 10 \cdot 0251 h_3 \\
 &\quad + 0 \cdot 1746 q_1 + 0 \cdot 0454 q_2 + 2 \cdot 4791 q_3
 \end{aligned}$$

37. The equations  $(p_2)$ ,  $(p_3)$  of the first group, and  $(q_2)$ ,  $(q_3)$  of the second were not formed, as our previous solution shewed that when  $\delta\epsilon$ ,  $\delta n$ ,  $\delta x_1$ , and  $\delta y_1$ , were eliminated the co-efficients of the remaining unknown quantities in these equations would be extremely small. It will be preferable to combine the equations  $(h_1)$ ,  $(h_2)$ ,  $(h_3)$ , and  $(k_1)$ ,  $(k_2)$ ,  $(k_3)$  before, instead of after, the elimination of  $\delta\epsilon$ ,  $\delta n$ ,  $\delta x_1$ , and  $\delta y_1$ , from them

If then we change the sign of the third equation in each group, and add it to the fourth and fifth, we obtain

$$\begin{aligned}
 111 \cdot 07 &= -17 \cdot 6009 \delta \epsilon + 6 \cdot 0448 \delta x_1 + 18 \cdot 2097 \delta x_2 \\
 &\quad - 6 \cdot 2965 h_1 + 13 \cdot 6704 h_2 + 19 \cdot 2974 h_3 \\
 &\quad - 13 \cdot 3971 p_1 - 15 \cdot 4639 p_2 + 2 \cdot 4144 p_3 \\
 194 \cdot 94 &= -11 \cdot 7320 \delta n + 3 \cdot 6520 \delta y_1 + 15 \cdot 0680 \delta y_2 \\
 &\quad + 0 \cdot 1951 h_1 + 7 \cdot 3294 h_2 + 14 \cdot 7069 h_3 \\
 &\quad - 0 \cdot 5785 q_1 - 0 \cdot 5496 q_2 + 2 \cdot 3663 q_3
 \end{aligned}$$

38. By means of ( $\epsilon$ ) and ( $n$ ) of Articles 35 and 36, eliminate  $\delta \epsilon$  and  $\delta n$  from ( $x$ ) and ( $y$ ), and also from the equations just found, and we have

$$\begin{aligned}
 (x) \ 202 \cdot 72 &= 6 \cdot 5294 \delta x_1 + 5 \cdot 4577 \delta x_2 + 3 \cdot 3234 h_1 + 8 \cdot 0291 h_2 \\
 &\quad + 6 \cdot 4240 h_3 + 1 \cdot 2659 p_1 + 0 \cdot 6477 p_2 + 5 \cdot 6529 p_3 \\
 (y) \ 42 \cdot 61 &= 1 \cdot 2578 \delta y_1 + 3 \cdot 3771 \delta y_2 + 0 \cdot 3822 h_1 + 2 \cdot 0739 h_2 \\
 &\quad + 3 \cdot 3916 h_3 + 0 \cdot 0836 q_1 + 0 \cdot 0298 q_2 + 0 \cdot 9513 q_3 \\
 268 \cdot 02 &= 11 \cdot 1297 \delta x_1 + 14 \cdot 4919 \delta x_2 + 5 \cdot 3967 h_1 + 14 \cdot 4642 h_2 \\
 &\quad + 15 \cdot 4449 h_3 + 2 \cdot 0175 p_1 + 1 \cdot 0266 p_2 + 9 \cdot 4702 p_3 \\
 168 \cdot 85 &= 5 \cdot 0833 \delta y_1 + 14 \cdot 8824 \delta y_2 + 1 \cdot 5261 h_1 + 8 \cdot 4906 h_2 \\
 &\quad + 14 \cdot 6979 h_3 + 0 \cdot 3320 q_1 + 0 \cdot 1189 q_2 + 3 \cdot 8313 q_3
 \end{aligned}$$

39. Substituting for  $\delta x_2$ ,  $\delta y_2$  their values in terms of  $\delta x_1$ ,  $\delta y_1$ , we find

$$\begin{aligned}
 6 \cdot 5294 \delta x_1 + 5 \cdot 4577 \delta x_2 &= 6 \cdot 5700 \delta x_1 + 0 \cdot 0490 \delta y_1 \\
 1 \cdot 2578 \delta y_1 + 3 \cdot 3771 \delta y_2 &= -0 \cdot 0303 \delta x_1 + 1 \cdot 2829 \delta y_1 \\
 11 \cdot 1297 \delta x_1 + 14 \cdot 4919 \delta x_2 &= 11 \cdot 2378 \delta x_1 + 0 \cdot 1300 \delta y_1 \\
 5 \cdot 0833 \delta y_1 + 14 \cdot 8824 \delta y_2 &= -0 \cdot 1335 \delta x_1 + 5 \cdot 1943 \delta y_1
 \end{aligned}$$

Hence, if we add to the two latter equations

$$- 1 \cdot 7106 (x) - 0 \cdot 03607 (y)$$

and  $0 \cdot 00165 (x) - 4 \cdot 0487 (y)$  respectively,

$\delta x_1$  and  $\delta y_1$  will be eliminated, and we shall obtain the following equations:

$$\begin{aligned}
 (1) \ 80 \cdot 28 &= 0 \cdot 2883 h_1 - 0 \cdot 7295 h_2 - 4 \cdot 4559 h_3 \\
 &\quad + 0 \cdot 0138 h_1 + 0 \cdot 0748 h_2 + 0 \cdot 1223 h_3 \\
 &\quad + 0 \cdot 1479 p_1 + 0 \cdot 0813 p_2 + 0 \cdot 1997 p_3 \\
 &\quad + 0 \cdot 0030 q_1 + 0 \cdot 0011 q_2 + 0 \cdot 0343 q_3 \\
 (2) \ 3 \cdot 34 &= -0 \cdot 0055 h_1 - 0 \cdot 0132 h_2 - 0 \cdot 0106 h_3 \\
 &\quad + 0 \cdot 0212 h_1 - 0 \cdot 0939 h_2 - 0 \cdot 0662 h_3 \\
 &\quad - 0 \cdot 0021 p_1 - 0 \cdot 0011 p_2 - 0 \cdot 0093 p_3 \\
 &\quad + 0 \cdot 0066 q_1 + 0 \cdot 0017 q_2 + 0 \cdot 0203 q_3
 \end{aligned}$$

40. Again, the equations of condition given by the Ancient Observations are

$$\begin{aligned}
 & \text{"} \\
 62 \cdot 6 = & \quad \delta\epsilon - 0 \cdot 8776 \quad \delta x_1 + 0 \cdot 5402 \quad \delta x_2 + 0 \cdot 7923 \quad h_1 + 0 \cdot 2554 \quad h_2 \\
 & - 39 \cdot 31 \quad \delta n - 0 \cdot 4795 \quad \delta y_1 + 0 \cdot 8415 \quad \delta y_2 + 0 \cdot 6101 \quad k_1 + 0 \cdot 9668 \quad k_2 \\
 & \quad - 0 \cdot 3875 \quad h_3 - 0 \cdot 9877 \quad p_1 - 0 \cdot 6870 \quad p_2 - 0 \cdot 1009 \quad p_3 \\
 & \quad + 0 \cdot 9219 \quad k_3 + 0 \cdot 1566 \quad q_1 - 0 \cdot 7267 \quad q_2 - 0 \cdot 9949 \quad q_3 \\
 \\
 84 \cdot 5 = & \quad \delta\epsilon + 0 \cdot 4975 \quad \delta x_1 - 0 \cdot 5050 \quad \delta x_2 - 0 \cdot 0887 \quad h_1 - 0 \cdot 9843 \quad h_2 \\
 & - 32 \cdot 30 \quad \delta n - 0 \cdot 8675 \quad \delta y_1 - 0 \cdot 8631 \quad \delta y_2 + 0 \cdot 9961 \quad k_1 - 0 \cdot 1767 \quad k_2 \\
 & \quad + 0 \cdot 2634 \quad h_3 - 0 \cdot 9085 \quad p_1 - 0 \cdot 3355 \quad p_2 + 0 \cdot 9681 \quad p_3 \\
 & \quad - 0 \cdot 9647 \quad k_3 - 0 \cdot 4178 \quad q_1 - 0 \cdot 9420 \quad q_2 - 0 \cdot 2506 \quad q_3 \\
 \\
 67 \cdot 2 = & \quad \delta\epsilon + 0 \cdot 6732 \quad \delta x_1 - 0 \cdot 0935 \quad \delta x_2 - 0 \cdot 2243 \quad h_1 - 0 \cdot 8994 \quad h_2 \\
 & - 31 \cdot 34 \quad \delta n - 0 \cdot 7394 \quad \delta y_1 - 0 \cdot 9956 \quad \delta y_2 + 0 \cdot 9745 \quad k_1 - 0 \cdot 4371 \quad k_2 \\
 & \quad + 0 \cdot 6277 \quad h_3 - 0 \cdot 8720 \quad p_1 - 0 \cdot 2815 \quad p_2 + 0 \cdot 9982 \quad p_3 \\
 & \quad - 0 \cdot 7785 \quad k_3 - 0 \cdot 4895 \quad q_1 - 0 \cdot 9596 \quad q_2 - 0 \cdot 0591 \quad q_3 \\
 \\
 - 51 \cdot 8 = & \quad \delta\epsilon - 0 \cdot 2616 \quad \delta x_1 - 0 \cdot 8631 \quad \delta x_2 - 0 \cdot 9436 \quad h_1 + 0 \cdot 7809 \quad h_2 \\
 & - 19 \cdot 59 \quad \delta n + 0 \cdot 9652 \quad \delta y_1 - 0 \cdot 5050 \quad \delta y_2 - 0 \cdot 3310 \quad k_1 + 0 \cdot 6247 \quad k_2 \\
 & \quad - 0 \cdot 5301 \quad h_3 - 0 \cdot 0731 \quad p_1 + 0 \cdot 3991 \quad p_2 - 0 \cdot 6801 \quad p_3 \\
 & \quad - 0 \cdot 8479 \quad k_3 - 0 \cdot 9973 \quad q_1 - 0 \cdot 9169 \quad q_2 + 0 \cdot 7331 \quad q_3 \\
 \\
 - 43 \cdot 2 = & \quad \delta\epsilon - 0 \cdot 4741 \quad \delta x_1 - 0 \cdot 5505 \quad \delta x_2 - 0 \cdot 8861 \quad h_1 + 0 \cdot 5704 \quad h_2 \\
 & - 18 \cdot 58 \quad \delta n + 0 \cdot 8805 \quad \delta y_1 - 0 \cdot 8348 \quad \delta y_2 - 0 \cdot 4634 \quad k_1 + 0 \cdot 8213 \quad k_2 \\
 & \quad - 0 \cdot 1248 \quad h_3 + 0 \cdot 0115 \quad p_1 + 0 \cdot 4532 \quad p_2 - 0 \cdot 8147 \quad p_3 \\
 & \quad - 0 \cdot 9922 \quad k_3 - 0 \cdot 9999 \quad q_1 - 0 \cdot 8914 \quad q_2 + 0 \cdot 5798 \quad q_3 \\
 \\
 - 50 \cdot 1 = & \quad \delta\epsilon - 0 \cdot 6430 \quad \delta x_1 - 0 \cdot 1731 \quad \delta x_2 - 0 \cdot 8191 \quad h_1 + 0 \cdot 3420 \quad h_2 \\
 & - 17 \cdot 68 \quad \delta n + 0 \cdot 7659 \quad \delta y_1 - 0 \cdot 9849 \quad \delta y_2 - 0 \cdot 5736 \quad k_1 + 0 \cdot 9397 \quad k_2 \\
 & \quad + 0 \cdot 2588 \quad h_3 + 0 \cdot 0871 \quad p_1 + 0 \cdot 5001 \quad p_2 - 0 \cdot 9063 \quad p_3 \\
 & \quad - 0 \cdot 9659 \quad k_3 - 0 \cdot 9962 \quad q_1 - 0 \cdot 8660 \quad q_2 + 0 \cdot 4225 \quad q_3 \\
 \\
 - 37 \cdot 8 = & \quad \delta\epsilon - 0 \cdot 9492 \quad \delta x_1 + 0 \cdot 8021 \quad \delta x_2 - 0 \cdot 5743 \quad h_1 - 0 \cdot 3404 \quad h_2 \\
 & - 15 \cdot 25 \quad \delta n + 0 \cdot 3145 \quad \delta y_1 - 0 \cdot 5972 \quad \delta y_2 - 0 \cdot 8186 \quad k_1 + 0 \cdot 9403 \quad k_2 \\
 & \quad + 0 \cdot 9652 \quad h_3 + 0 \cdot 2872 \quad p_1 + 0 \cdot 6192 \quad p_2 - 0 \cdot 9984 \quad p_3 \\
 & \quad - 0 \cdot 2613 \quad k_3 - 0 \cdot 9579 \quad q_1 - 0 \cdot 7852 \quad q_2 - 0 \cdot 0560 \quad q_3 \\
 \\
 - 20 \cdot 5 = & \quad \delta\epsilon - 0 \cdot 9985 \quad \delta x_1 + 0 \cdot 9942 \quad \delta x_2 - 0 \cdot 3671 \quad h_1 - 0 \cdot 7304 \quad h_2 \\
 & - 13 \cdot 60 \quad \delta n - 0 \cdot 0538 \quad \delta y_1 + 0 \cdot 1074 \quad \delta y_2 - 0 \cdot 9302 \quad k_1 + 0 \cdot 6830 \quad k_2 \\
 & \quad + 0 \cdot 9035 \quad h_3 + 0 \cdot 4164 \quad p_1 + 0 \cdot 6928 \quad p_2 - 0 \cdot 9251 \quad p_3 \\
 & \quad + 0 \cdot 4286 \quad k_3 - 0 \cdot 9092 \quad q_1 - 0 \cdot 7212 \quad q_2 - 0 \cdot 3796 \quad q_3 \\
 \\
 - 2 \cdot 4 = & \quad \delta\epsilon - 0 \cdot 9633 \quad \delta x_1 + 0 \cdot 8560 \quad \delta x_2 - 0 \cdot 2363 \quad h_1 - 0 \cdot 8883 \quad h_2 \\
 & - 12 \cdot 64 \quad \delta n - 0 \cdot 2684 \quad \delta y_1 + 0 \cdot 5170 \quad \delta y_2 - 0 \cdot 9717 \quad k_1 + 0 \cdot 4593 \quad k_2 \\
 & \quad + 0 \cdot 6562 \quad h_3 + 0 \cdot 4882 \quad p_1 + 0 \cdot 7327 \quad p_2 - 0 \cdot 8345 \quad p_3 \\
 & \quad + 0 \cdot 7546 \quad k_3 - 0 \cdot 8727 \quad q_1 - 0 \cdot 6806 \quad q_2 - 0 \cdot 5511 \quad q_3
 \end{aligned}$$

41. The equation for finding  $p_s$  may be formed, as before, by multiplying the above equations taken in order by

$$-0 \cdot 8, -0 \cdot 6, +1 \cdot 0, +1 \cdot 0, +0 \cdot 9, +0 \cdot 6, +0 \cdot 4, +0 \cdot 3$$

beginning with the second; and the equation for  $q_s$  by multiplying the same equations by

$$1 \cdot 0, 1 \cdot 0, 0 \cdot 5, 0 \cdot 4, 0 \cdot 3, 0 \cdot 2, 0 \cdot 1, 0 \cdot 1.$$

Thus we obtain

$$\begin{aligned} -279 \cdot 64 = & 2 \cdot 80 \delta \epsilon - 3 \cdot 3742 \delta x_1 + 0 \cdot 0265 \delta x_2 - 2 \cdot 9237 h_1 + 2 \cdot 2232 h_2 \\ & - 27 \cdot 82 \delta n + 3 \cdot 7593 \delta y_1 - 1 \cdot 0986 \delta y_2 - 3 \cdot 8471 k_1 + 3 \cdot 6706 k_2 \\ & + 0 \cdot 1281 h_3 + 1 \cdot 7522 p_1 + 2 \cdot 6081 p_2 - 4 \cdot 9033 p_3 \\ & - 1 \cdot 2295 h_3 - 3 \cdot 4661 q_1 - 2 \cdot 2221 q_2 + 1 \cdot 5785 q_3 \end{aligned}$$

$$\begin{aligned} 83 \cdot 56 = & 3 \cdot 60 \delta \epsilon + 0 \cdot 2714 \delta x_1 - 0 \cdot 9567 \delta x_2 - 1 \cdot 5602 h_1 - 1 \cdot 3924 h_2 \\ & - 91 \cdot 84 \delta n - 0 \cdot 5116 \delta y_1 - 2 \cdot 7976 \delta y_2 + 1 \cdot 0937 k_1 + 0 \cdot 6112 k_2 \\ & + 1 \cdot 0027 h_3 - 1 \cdot 6385 p_1 + 0 \cdot 1802 p_2 + 0 \cdot 6529 p_3 \\ & - 2 \cdot 7879 h_3 - 2 \cdot 4746 q_1 - 3 \cdot 2736 q_2 + 0 \cdot 3113 q_3 \end{aligned}$$

42. Eliminate  $\delta \epsilon$  and  $\delta n$  by means of  $(\epsilon)$  and  $(n)$  of Articles 35 and 36, and these equations become

$$\begin{aligned} -361 \cdot 72 = & -4 \cdot 1831 \delta x_1 + 0 \cdot 6179 \delta x_2 - 4 \cdot 7839 h_1 + 2 \cdot 0969 h_2 \\ & + 7 \cdot 1533 \delta y_1 - 1 \cdot 5388 \delta y_2 - 0 \cdot 6909 k_1 + 6 \cdot 4242 k_2 \\ & + 0 \cdot 7410 h_3 - 0 \cdot 7000 p_1 - 0 \cdot 0153 p_2 - 6 \cdot 0258 p_3 \\ & - 1 \cdot 2508 h_3 - 1 \cdot 3068 q_1 - 0 \cdot 6369 q_2 + 5 \cdot 0525 q_3 \end{aligned}$$

$$\begin{aligned} -146 \cdot 69 = & -0 \cdot 7686 \delta x_1 - 0 \cdot 1963 \delta x_2 - 3 \cdot 9519 h_1 - 1 \cdot 5548 h_2 \\ & + 10 \cdot 6926 \delta y_1 - 4 \cdot 2508 \delta y_2 + 11 \cdot 5128 k_1 + 9 \cdot 7013 k_2 \\ & + 1 \cdot 7907 h_3 - 4 \cdot 7913 p_1 - 3 \cdot 1927 p_2 - 0 \cdot 7902 p_3 \\ & - 2 \cdot 8583 h_3 + 4 \cdot 6536 q_1 + 1 \cdot 9595 q_2 + 11 \cdot 7796 q_3 \end{aligned}$$

43. Substituting for  $\delta x_2, \delta y_2$ , their values in terms of  $\delta x_1, \delta y_1$ , we find

$$\begin{aligned} -4 \cdot 1831 \delta x_1 + 7 \cdot 1533 \delta y_1 + 0 \cdot 6179 \delta x_2 - 1 \cdot 5388 \delta y_2 = & -4 \cdot 1647 \delta x_1 + 7 \cdot 1473 \delta y_1 \\ -0 \cdot 7686 \delta x_1 + 10 \cdot 6926 \delta y_1 - 0 \cdot 1963 \delta x_2 - 4 \cdot 2508 \delta y_2 = & -0 \cdot 7319 \delta x_1 + 10 \cdot 6591 \delta y_1 \end{aligned}$$

Hence, if to the equations just found, we add

$$+0 \cdot 60808 (x) - 5 \cdot 5942 (y)$$

and

$$+0 \cdot 07306 (x) - 8 \cdot 3110 (y) \text{ respectively,}$$

$\delta x_1$  and  $\delta y_1$  will be eliminated, and we shall obtain the following equations :

$$\begin{aligned} (3) \quad -476 \cdot 84 = & -2 \cdot 7630 h_1 + 6 \cdot 9793 h_2 + 4 \cdot 6473 h_3 \\ & - 2 \cdot 8290 k_1 - 5 \cdot 1777 k_2 - 20 \cdot 2242 k_3 \\ & + 0 \cdot 0698 p_1 + 0 \cdot 3785 p_2 - 2 \cdot 5884 p_3 \\ & - 1 \cdot 7748 q_1 - 0 \cdot 8036 q_2 - 0 \cdot 2693 q_3 \end{aligned}$$

$$\begin{aligned}
 (4) \quad -486^{\cdot}03 &= -3\cdot7091 h_1 - 0\cdot9682 h_2 + 2\cdot2600 h_3 \\
 &\quad + 8\cdot3364 k_1 - 7\cdot5348 k_2 - 31\cdot0457 k_3 \\
 &\quad - 4\cdot6988 p_1 - 3\cdot1454 p_2 - 0\cdot3772 p_3 \\
 &\quad + 3\cdot9584 q_1 + 1\cdot7118 q_2 + 3\cdot8734 q_3
 \end{aligned}$$

44. Eliminate the left hand members from equations (2), (3), and (4) of Articles 39 and 43, by means of equation (1), and we have

$$\begin{aligned}
 0 &= 0\cdot4200 h_1 - 0\cdot4114 h_2 - 4\cdot2014 h_3 + 0\cdot1980 p_1 + 0\cdot1069 p_2 + 0\cdot4236 p_3 \\
 &\quad - 0\cdot4964 k_1 + 2\cdot3306 k_2 + 23\cdot3213 k_3 - 0\cdot1567 q_1 - 0\cdot0409 q_2 - 0\cdot4531 q_3 \\
 0 &= -1\cdot0507 h_1 + 2\cdot6465 h_2 - 21\cdot8182 h_3 + 0\cdot9482 p_1 + 0\cdot8614 p_2 - 1\cdot4023 p_3 \\
 &\quad - 2\cdot7471 k_1 - 4\cdot7334 k_2 - 19\cdot4976 k_3 - 1\cdot7569 q_1 - 0\cdot7972 q_2 - 0\cdot0655 q_3 \\
 0 &= -1\cdot9638 h_1 - 5\cdot3845 h_2 - 24\cdot7155 h_3 - 3\cdot8034 p_1 - 2\cdot6532 p_2 + 0\cdot8317 p_3 \\
 &\quad + 8\cdot4199 k_1 - 7\cdot0819 k_2 - 30\cdot3051 k_3 + 3\cdot9767 q_1 + 1\cdot7183 q_2 + 4\cdot0811 q_3
 \end{aligned}$$

45. If, as before, we put  $\varepsilon - \varepsilon' = \theta$  and  $\varepsilon - \varpi = \beta$ , it may be seen that

$$\begin{aligned}
 \frac{h_1}{m'} &= -42^{\cdot}33 \sin \theta, & \frac{h_2}{m'} &= 76^{\cdot}55 \sin 2\theta \\
 \frac{k_1}{m'} &= -42^{\cdot}33 \cos \theta, & \frac{k_2}{m'} &= 76^{\cdot}55 \cos 2\theta \\
 \frac{h_3}{m'} &= 7^{\cdot}25 \sin 3\theta + 0\cdot007460 \frac{p_3}{m'} + 0\cdot008974 \frac{q_3}{m'} \\
 \frac{k_3}{m'} &= 7^{\cdot}25 \cos 3\theta - 0\cdot008974 \frac{p_3}{m'} + 0\cdot007460 \frac{q_3}{m'} \\
 \frac{p_1}{m'} &= 0\cdot20 \sin (\theta - \beta) - 0\cdot074738 \left\{ \frac{p_3}{m'} \cos 2\theta - \frac{q_3}{m'} \sin 2\theta \right\} \\
 \frac{q_1}{m'} &= -0\cdot20 \cos (\theta - \beta) + 0\cdot074738 \left\{ \frac{p_3}{m'} \sin 2\theta + \frac{q_3}{m'} \cos 2\theta \right\} \\
 \frac{p_2}{m'} &= 32^{\cdot}91 \sin (2\theta - \beta) + 0\cdot259765 \left\{ \frac{p_3}{m'} \cos \theta - \frac{q_3}{m'} \sin \theta \right\} \\
 \frac{q_2}{m'} &= 32^{\cdot}91 \cos (2\theta - \beta) + 0\cdot259765 \left\{ \frac{p_3}{m'} \sin \theta + \frac{q_3}{m'} \cos \theta \right\}
 \end{aligned}$$

46. Substituting these expressions in the above equations, and putting for  $\beta$  its value  $50^{\circ} 15' \cdot 8$ , we obtain

$$\begin{aligned}
 0 &= -(1\cdot24872) \sin \theta + (1\cdot32231) \cos \theta - (1\cdot48110) \sin 2\theta + (2\cdot24265) \cos 2\theta \\
 &\quad - (1\cdot48373) \sin 3\theta + (2\cdot22809) \cos 3\theta + (9\cdot26254) \frac{p_3}{m'} - (9\cdot50079) \frac{q_3}{m'} \\
 &\quad + (8\cdot44376) \left\{ \frac{p_3}{m'} \cos \theta - \frac{q_3}{m'} \sin \theta \right\} - (8\cdot02630) \left\{ \frac{p_3}{m'} \sin \theta + \frac{q_3}{m'} \cos \theta \right\} \\
 &\quad - (8\cdot17031) \left\{ \frac{p_3}{m'} \cos 2\theta - \frac{q_3}{m'} \sin 2\theta \right\} - (8\cdot06861) \left\{ \frac{p_3}{m'} \sin 2\theta + \frac{q_3}{m'} \cos 2\theta \right\}
 \end{aligned}$$

$$\begin{aligned}
 0 = & (1 \cdot 65190) \sin \theta + (2 \cdot 06584) \cos \theta + (2 \cdot 30220) \sin 2\theta - (2 \cdot 60306) \cos 2\theta \\
 & - (2 \cdot 19916) \sin 3\theta - (2 \cdot 15032) \cos 3\theta - (0 \cdot 14305) \frac{p_s}{m'} - (9 \cdot 60933) \frac{q_s}{m'} \\
 & + (9 \cdot 34981) \left\{ \frac{p_s}{m'} \cos \theta - \frac{q_s}{m'} \sin \theta \right\} - (9 \cdot 31615) \left\{ \frac{p_s}{m'} \sin \theta + \frac{q_s}{m'} \cos \theta \right\} \\
 & - (8 \cdot 85046) \left\{ \frac{p_s}{m'} \cos 2\theta - \frac{q_s}{m'} \sin 2\theta \right\} - (9 \cdot 11828) \left\{ \frac{p_s}{m'} \sin 2\theta + \frac{q_s}{m'} \cos 2\theta \right\} \\
 0 = & (1 \cdot 91407) \sin \theta - (2 \cdot 55189) \cos \theta - (2 \cdot 62790) \sin 2\theta - (2 \cdot 64230) \cos 2\theta \\
 & - (2 \cdot 25331) \sin 3\theta - (2 \cdot 34185) \cos 3\theta + (9 \cdot 96344) \frac{p_s}{m'} + (0 \cdot 56029) \frac{q_s}{m'} \\
 & - (9 \cdot 83835) \left\{ \frac{p_s}{m'} \cos \theta - \frac{q_s}{m'} \sin \theta \right\} + (9 \cdot 64968) \left\{ \frac{p_s}{m'} \sin \theta + \frac{q_s}{m'} \cos \theta \right\} \\
 & + (9 \cdot 45371) \left\{ \frac{p_s}{m'} \cos 2\theta - \frac{q_s}{m'} \sin 2\theta \right\} + (9 \cdot 47306) \left\{ \frac{p_s}{m'} \sin 2\theta + \frac{q_s}{m'} \cos 2\theta \right\}
 \end{aligned}$$

where the numbers enclosed within parentheses denote the logarithms of the corresponding co-efficients, as before.

47. From these equations, we find, by the same method as before

$$\theta = -46^\circ 55', \quad \frac{p_s}{m'} = 138'' \cdot 92, \quad \frac{q_s}{m'} = -109'' \cdot 83$$

Hence, since  $\epsilon = 217^\circ 55'$ ,  $\epsilon' = 264^\circ 50'$ , the mean longitude of the disturbing planet at the epoch 1810·328. The sidereal motion in 36 synodic periods of URANUS  $= 57^\circ 42'$ , Precession  $= 30'$ .  $\therefore$  Mean Longitude at the time 1846·762, or October 6, 1846,  $= 323^\circ 2'$ .

Also, the expressions for  $\frac{p_s}{m'}$  and  $\frac{q_s}{m'}$  are

$$\begin{aligned}
 \frac{p_s}{m'} &= 33'' \cdot 93 \sin(3\theta - \beta) - 63'' \cdot 41 e' \sin(3\theta - \beta') \\
 \frac{q_s}{m'} &= 33'' \cdot 93 \cos(3\theta - \beta) - 63'' \cdot 41 e' \cos(3\theta - \beta')
 \end{aligned}$$

where  $\epsilon - \omega' = \beta'$ .

Equating these to the values given above, we find  $e' = 2 \cdot 4123$ ,  $\beta' = 279^\circ 14'$ , and  $\therefore \omega' = 298^\circ 41'$ . Hence longitude of the perihelion in 1846  $= 299^\circ 11'$ .

Lastly, substituting the values just obtained in equation (1) of Article 39, we find  $m' = 0 \cdot 75017$ .

48. Hence the values of the mass and elements of the orbit of the disturbing planet, resulting from the second hypothesis as to the mean distance, are the following:

$$\frac{a}{a'} = 0 \cdot 515$$

Mean Longitude of the Planet, October 6, 1846,	323° 2'
Longitude of the Perihelion - - - - -	299 11
Eccentricity of the Orbit - - - - -	0·120615
Mass (that of the SUN being 1) - - - - -	0·00015003 .

49. From the values of  $m'$ ,  $\theta$ ,  $\frac{P_2}{m'}$  and  $\frac{q_2}{m'}$ , found above, the values of the quantities  $h$ ,  $k$ ,  $p$ , and  $q$ , corresponding to each hypothesis, are immediately determined. Thus we find

*1st Hypothesis.*

$$\frac{a}{a'} = 0.5$$

$$\begin{array}{ll} h_1 = 23.98 & k_1 = -19.07 \\ h_2 = -47.58 & k_2 = -11.00 \\ h_3 = -1.93 & k_3 = -7.64 \\ p_1 = 9.93 & q_1 = -8.31 \\ p_2 = -8.54 & q_2 = -55.36 \\ p_3 = 224.90 & q_3 = -171.63 \end{array}$$

*2nd Hypothesis.*

$$\frac{a}{a'} = 0.515$$

$$\begin{array}{ll} h_1 = 23.19 & k_1 = -21.69 \\ h_2 = -57.30 & k_2 = -3.83 \\ h_3 = -3.40 & k_3 = -5.76 \\ p_1 = 6.52 & q_1 = -7.34 \\ p_2 = -11.62 & q_2 = -54.39 \\ p_3 = 104.21 & q_3 = -82.39 \end{array}$$

50. And by substituting these values in the equations ( $\epsilon$ ), ( $n$ ), ( $x$ ), and ( $y$ ), we obtain

*1st Hypothesis.*

$$\frac{a}{a'} = 0.5$$

$$\begin{array}{ll} \delta\epsilon = -49.77 & \delta n = -0.702 \\ \delta x_1 = -130.69 & \delta y_1 = 222.38 \\ \delta x_2 = 1.02 & \delta y_2 = 2.83 \end{array}$$

*2nd Hypothesis.*

$$\frac{a}{a'} = 0.515$$

$$\begin{array}{ll} \delta\epsilon = -43.23 & \delta n = -0.5417 \\ \delta x_1 = 1.77 & \delta y_1 = 123.98 \\ \delta x_2 = 1.13 & \delta y_2 = 0.91 \end{array}$$

and the corresponding corrections of the elliptic elements will be

$$\frac{\delta a}{a} = 0.00000999$$

$$\begin{array}{l} \delta e = 20.83 \\ e\delta\omega = 127.27 \end{array}$$

$$\frac{\delta a}{a} = 0.00000771$$

$$\begin{array}{l} \delta e = 40.31 \\ e\delta\omega = 47.10 \end{array}$$

It will be seen that the corrections of the eccentricity and longitude of perihelion vary very rapidly with a change in the assumed mean distance.

51. If these quantities be substituted in the expressions before given, we obtain the following theoretical corrections of the mean longitude, each of these corrections being divided into two parts, of which the first is due to the changes in the elements of the orbit of Uranus, and the second to the action of the disturbing planet.

## HYPOTHESIS I.

*Ancient Observations.*

Year.	"	"	"
1712	-288.0	+365.8	=+77.8
1715	-283.1	+357.1	=+74.0
1750	+210.5	-260.7	=-50.2
1753	+218.1	-267.0	=-48.9
1756	+214.0	-260.0	=-46.0
1764	+154.0	-186.7	=-32.7
1769	+79.6	-100.7	=-21.1
1771	+27.6	-41.8	=-14.2

*Modern Observations.*

Year.	"	"	"
1780	-126.12	+129.27	=+3.15
1783	-180.28	+188.70	=+8.42
1786	-227.66	+240.36	=+12.70
1789	-265.70	+281.63	=+15.93
1792	-292.25	+310.38	=+18.13
1795	-305.84	+325.27	=+19.43
1798	-305.67	+325.72	=+20.05
1801	-291.77	+312.05	=+20.28
1804	-264.95	+285.38	=+20.43
1807	-226.78	+247.51	=+20.73
1810	-179.43	+200.76	=+21.33
1813	-125.59	+147.72	=+22.13
1816	-68.21	+91.02	=+22.81
1819	-10.40	+33.18	=+22.78
1822	+44.84	-23.64	=+21.20
1825	+94.69	-77.64	=+17.05
1828	+136.73	-127.48	=+9.25
1831	+168.94	-172.17	=-3.23
1834	+189.85	-211.04	=-21.19
1837	+198.51	-243.59	=-45.08
1840	+194.54	-269.36	=-74.82

## HYPOTHESIS II.

*Ancient Observations.*

Year.	"	"	"
1712	-133.7	+211.9	=+78.2
1715	-117.7	+191.5	=+73.8
1750	+85.2	-134.4	=-49.2
1753	+73.8	-122.2	=-48.4
1756	+59.1	-105.2	=-46.1
1764	+2.7	-36.4	=-33.7
1769	-43.1	+20.8	=-22.3
1771	-69.9	+54.7	=-15.2

*Modern Observations.*

Year.	"	"	"
1780	-133.10	+135.98	=+2.88
1783	-149.47	+157.87	=+8.40
1786	-160.15	+172.99	=+12.84
1789	-164.52	+180.64	=+16.12
1792	-162.30	+180.58	=+18.28
1795	-153.59	+173.07	=+19.48
1798	-138.87	+158.86	=+19.99
1801	-118.95	+139.08	=+20.13
1804	-94.96	+115.21	=+20.25
1807	-68.25	+88.85	=+20.60
1810	-40.33	+61.61	=+21.28
1813	-12.72	+34.91	=+22.19
1816	+13.08	+9.88	=+22.96
1819	+35.71	-12.74	=+22.97
1822	+54.04	-32.68	=+21.36
1825	+67.18	-50.08	=+17.10
1828	+74.52	-65.37	=+9.15
1831	+75.74	-79.21	=-3.47
1834	+70.85	-92.31	=-21.46
1837	+60.08	-105.25	=-45.17
1840	+43.98	-118.38	=-74.40



52. Comparing these with the corrections of mean longitude derived from observation, we find the remaining differences to be the following :

<i>Ancient Observations.</i>			<i>Modern Observations.</i>		
Year.	Observation—Theory.		Year.	Observation—Theory.	
	Hypoth. I.	Hypoth. II.		Hypoth. I.	Hypoth. II.
1712	+ 6 <sup>"</sup> ·7	+ 6 <sup>"</sup> ·3	1780	+ 0 <sup>"</sup> ·27	+ 0 <sup>"</sup> ·54
1715	— 6 <sup>"</sup> ·8	— 6 <sup>"</sup> ·6	1783	— 0 <sup>"</sup> ·23	— 0 <sup>"</sup> ·21
1750	— 1 <sup>"</sup> ·6	— 2 <sup>"</sup> ·6	1786	— 0 <sup>"</sup> ·96	— 1 <sup>"</sup> ·10
1753	+ 5 <sup>"</sup> ·7	+ 5 <sup>"</sup> ·2	1789	+ 1 <sup>"</sup> ·82	+ 1 <sup>"</sup> ·63
1756	— 4 <sup>"</sup> ·1	— 4 <sup>"</sup> ·0	1792	— 0 <sup>"</sup> ·91	— 1 <sup>"</sup> ·06
1764	— 5 <sup>"</sup> ·1	— 4 <sup>"</sup> ·1	1795	+ 0 <sup>"</sup> ·09	+ 0 <sup>"</sup> ·04
1769	+ 0 <sup>"</sup> ·6	+ 1 <sup>"</sup> ·8	1798	— 0 <sup>"</sup> ·99	— 0 <sup>"</sup> ·93
1771	+ 11 <sup>"</sup> ·8	+ 12 <sup>"</sup> ·8	1801	— 0 <sup>"</sup> ·04	+ 0 <sup>"</sup> ·11
			1804	+ 1 <sup>"</sup> ·76	+ 1 <sup>"</sup> ·94
			1807	— 0 <sup>"</sup> ·21	— 0 <sup>"</sup> ·08
			1810	+ 0 <sup>"</sup> ·56	+ 0 <sup>"</sup> ·61
			1813	— 0 <sup>"</sup> ·94	— 1 <sup>"</sup> ·00
			1816	— 0 <sup>"</sup> ·31	— 0 <sup>"</sup> ·46
			1819	— 2 <sup>"</sup> ·00	— 2 <sup>"</sup> ·19
			1822	+ 0 <sup>"</sup> ·30	+ 0 <sup>"</sup> ·14
			1825	+ 1 <sup>"</sup> ·92	+ 1 <sup>"</sup> ·87
			1828	+ 2 <sup>"</sup> ·25	+ 2 <sup>"</sup> ·35
			1831	— 1 <sup>"</sup> ·06	— 0 <sup>"</sup> ·82
			1834	— 1 <sup>"</sup> ·44	— 1 <sup>"</sup> ·17
			1837	— 1 <sup>"</sup> ·62	— 1 <sup>"</sup> ·53
			1840	+ 1 <sup>"</sup> ·73	+ 1 <sup>"</sup> ·31

The largest difference in the above table, *viz.*, that for 1771, is deduced from a single observation ; whereas the difference immediately preceding it, which is deduced from the mean of several, is very small.

53. The results of the two theories agree very closely with each other, and with observation, till we come to the later years of the series ; and it is to be observed that the difference between the theories becomes sensible at precisely the point where they both show symptoms of diverging from the observations, the errors of the second hypothesis, however, being less than those of the other.

Recent observations show that the errors of the theory soon become very sensible, though decidedly less for the second hypothesis than for the first. The following are the differences of mean longitude as deduced from theory and observation, for the oppositions of 1843, 1844, and 1845 :

Year.	Observation—Theory.	
	Hypoth. I.	Hypoth. II.
1843	+ 7 <sup>"</sup> ·11	+ 5 <sup>"</sup> ·77
1844	+ 8 <sup>"</sup> ·79	+ 7 <sup>"</sup> ·05
1845	+ 12 <sup>"</sup> ·40	+ 10 <sup>"</sup> ·18

For the observations of the last two years, I am indebted to the kindness of the Astronomer Royal. The three years nearly agree in showing that the errors of the first hypothesis are to those of the second in the ratio of 5 to 4, from which I inferred, in a letter to the Astronomer Royal, dated September 2, 1846, that the assumption of  $\frac{a}{a'} = \sin 35^\circ = 0.574$ , would probably satisfy all the observations very nearly.

54. The results which I have deduced from Professor Challis's observations of the planet, strongly confirm the inference that the mean distance should be considerably diminished. It is of course impossible to determine precisely, without actual calculation, the alteration in longitude which would be produced by such a diminution in the distance. By comparing the values of  $\theta$  given by the two hypotheses, it may be seen, however, that if we took successively smaller and smaller values for the mean distance, the values found for the mean longitude in 1810 would probably go on diminishing, while at the same time the mean motion from 1810 to 1846 would rapidly increase, so that the corresponding values of the mean longitude at the present time would probably soon arrive at a minimum, and afterwards begin again to increase. This I believe to be the reason why the longitude found on the supposition of too large a value for the mean distance agrees so nearly with observation. In consequence of not making sufficient allowance for the increase in the mean motion, I hastily inferred, in my letter to the Astronomer Royal mentioned above, that the effect of a diminution in the mean distance would be to diminish the mean longitude.

55. I have already mentioned that I thought it unsafe to employ Flamsteed's observation of 1690 in forming the equations of condition, as the interval between it and all the others is so large. The difference between it and the theory appears to be very considerable, and greater for the second hypothesis than for the first, the errors being  $+44''.5$  and  $+50''.0$  respectively. These errors would probably be increased by diminishing the mean distance. It would be desirable that Flamsteed's manuscripts should be examined with reference to this point.

56. The corrections of the Tabular Radius Vector of Uranus may be easily deduced from those of the mean longitude by means of the following formula:

$$\begin{aligned} \frac{\delta r}{r} = & \frac{1}{r} \frac{dr}{d\epsilon} \delta\zeta - \frac{1}{2n} \frac{d}{dt} \delta\zeta + \frac{1}{4} \frac{\delta a}{a} - \frac{1}{2} \frac{e \delta e}{1-e^2} - \frac{1}{6} m' a^2 \frac{dA_0}{da} \\ & + \frac{m'}{2} \Sigma C_i \cos i \{nt - n't + \epsilon - \epsilon'\} \\ & + m'e \Sigma D_i \cos i \{i(nt - n't + \epsilon - \epsilon') - nt - \epsilon + \omega\} \\ & + m'e' \Sigma E_i \cos i \{i(nt - n't + \epsilon - \epsilon') - nt - \epsilon + \omega'\} \end{aligned}$$

where  $\delta\zeta$  denotes the whole correction of the mean longitude at the time  $t$ ,

$$\frac{1}{r} \frac{dr}{d\epsilon} = e \sin \{nt + \epsilon - \omega\} + \frac{3e^3}{2} \sin 2 \{nt + \epsilon - \omega\} \text{ nearly,}$$

$$C_i = \frac{1}{2} \frac{\pi}{n - n'} a A'$$

$$D_i = -\frac{1}{4} \frac{i n}{i(n-n')-n} \left\{ 2 i a A_i + a^2 \frac{d A_i}{d a} \right\}$$

$$E_i = \frac{1}{4} \frac{(i-1) n}{i(n-n')-n} \left\{ (2 i-1) a A_{i-1} + a^2 \frac{d A_{i-1}}{d a} \right\}$$

i assuming all integral values positive and negative not including zero.

57. By substituting in this formula the values of  $m'$ ,  $\delta a$ ,  $\delta e$ , &c., already obtained, and putting  $a = 19 \cdot 191$ , we find the following results corresponding to the two assumed values of the mean distance.

#### HYPOTHESIS I.

$$\frac{a}{r} \delta r = \frac{a}{r} \frac{dr}{d\epsilon} \delta \zeta - \frac{a}{2} \frac{d \delta \zeta}{ndt} = 0 \cdot 000089$$

$$+ 0 \cdot 000069 \cos \{nt - n't + \epsilon - \epsilon'\}$$

$$+ 0 \cdot 000259 \cos 2 \{nt - n't + \epsilon - \epsilon'\}$$

$$+ 0 \cdot 000109 \cos 3 \{nt - n't + \epsilon - \epsilon'\}$$

$$+ 0 \cdot 000016 \cos \{n't + \epsilon' - \varpi\}$$

$$- 0 \cdot 000168 \cos \{nt - 2n't + \epsilon - 2\epsilon' + \varpi\}$$

$$+ 0 \cdot 000078 \cos \{nt - 2n't + \epsilon - 2\epsilon' + \varpi'\}$$

$$- 0 \cdot 000049 \cos \{2nt - 3n't + 2\epsilon - 3\epsilon' + \varpi\}$$

$$+ 0 \cdot 000209 \cos \{2nt - 3n't + 2\epsilon - 3\epsilon' + \varpi'\}$$

#### HYPOTHESIS II.

$$\frac{a}{r} \delta r = \frac{a}{r} \frac{dr}{d\epsilon} \delta \zeta - \frac{a}{2} \frac{d \delta \zeta}{ndt} = 0 \cdot 000144$$

$$+ 0 \cdot 000073 \cos \{nt - n't + \epsilon - \epsilon'\}$$

$$+ 0 \cdot 000266 \cos 2 \{nt - n't + \epsilon - \epsilon'\}$$

$$+ 0 \cdot 000115 \cos 3 \{nt - n't + \epsilon - \epsilon'\}$$

$$+ 0 \cdot 000016 \cos \{n't + \epsilon' - \varpi\}$$

$$- 0 \cdot 000188 \cos \{nt - 2n't + \epsilon - 2\epsilon' + \varpi\}$$

$$+ 0 \cdot 000068 \cos \{nt - 2n't + \epsilon - 2\epsilon' + \varpi'\}$$

$$- 0 \cdot 000053 \cos \{2nt - 3n't + 2\epsilon - 3\epsilon' + \varpi\}$$

$$+ 0 \cdot 000165 \cos \{2nt - 3n't + 2\epsilon - 3\epsilon' + \varpi'\}$$

58. The values of  $\delta \zeta$  and  $\frac{d \delta \zeta}{dt}$  for several late years, are the following:

#### HYPOTHESIS I.

Year.	$\delta \zeta$ "	$\frac{d \delta \zeta}{dt}$ "
1834	— 21 · 19	— 20 · 93
1840	— 74 · 82	— 32 · 34
1846	— 148 · 65	— 39 · 94

#### HYPOTHESIS II.

	"	"
1834	— 21 · 46	— 20 · 85
1840	— 74 · 40	— 31 · 62
1846	— 145 · 91	— 38 · 30

Hence, by means of the above formulæ, we find the corrections of the tabular radius vector, to be

Year.	Hypothesis I.	Hypothesis II.
1834	+0 '00505	+0 '00492
1840	+0 '00722	+0 '00696
1846	+0 '00868	+0 '00825

59. By far the most important part of these corrections arises from the term  $-\frac{1}{2} r \frac{d\delta\zeta}{ndt}$ , and may therefore be immediately deduced from a comparison of the observed angular motion of Uranus with that given by the Tables. In fact, the corrections given by this term alone for the epochs above-mentioned, are

Year.	Hypothesis I.	Hypothesis II.
1834	+0 '00447	+0 '00445
1840	+0 '00694	+0 '00678
1846	+0 '00853	+0 '00818

which, as we see, differ very little from the complete values just found. The correction for 1834, very nearly agrees with that which Mr. Airy has deduced from observation in the *Astronomische Nachrichten*. The corrections for subsequent years are rather larger than those given by the Greenwich Observations, the results of the second hypothesis being, as in the case of the longitude, nearer the truth than those of the first.

60. I made some attempts, by discussing the observations of latitude, to find approximate values of the longitude of the node and inclination of the orbit of the disturbing planet, but the results were not satisfactory. The perturbations of the latitude are in fact exceedingly small, and during the comparatively short period of three-fourths of a revolution, are nearly confounded with the effects of a constant alteration in the inclination and the position of the node of URANUS, so that very small errors in the observations may entirely vitiate the result.

61. The perturbations of Saturn produced by the new planet, though small, will still be sensible, and it would be interesting to inquire whether, if they were taken into account, the values of the masses of Jupiter and Uranus found from their action on Saturn would be more consistent with those determined by other means, than they appear to be at present. The reduction of the Greenwich Planetary Observations renders such an inquiry comparatively easy, and it is to be hoped that English astronomers will not be the last to avail themselves of the treasures of observation thus laid open to the world.

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